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# Bohr Density of Simple Linear Group Orbits*

François Ziegler (GSU)

April 12, 2013

*http://arxiv.org/abs/1211.3783, joint with Roger Howe (Yale).

# Introduction

Let

- $G$ : a noncompact, simple Lie group
- $V$ : a finite-dimensional, irreducible $G$-module over $\mathbf{R}$
- $V^*$ : the dual module ($= \mathrm{Hom}(V, \mathbf{R})$ with contragredient $G$-action)
- $bV$ : the Bohr compactification of $V$ ($=$ dual group of $(V^*_{\mathrm{discrete}}, +)$).

We have a dense inclusion $V \hookrightarrow bV$ ($=$ dual map of $V^*_{\mathrm{discrete}} \to V^*$).

**Very "thin" subsets of $V$ can be already dense in $bV$.** Indeed, we shall prove:

**Theorem**

Every nonzero $G$-orbit in $V$ is dense in $bV$.

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Let $\mathcal{O}$ be a subset of V. Then the following are equivalent:

1. $\mathcal{O}$ is dense in $b$V;
2. $\alpha(\mathcal{O})$ is dense in $\alpha(\mathrm{V})$ whenever $\alpha$ is a continuous morphism from V to a compact topological group;
3. Almost periodic functions on V are determined by their restriction to $\mathcal{O}$;
4. Haar measure $\eta$ on $b$V is the weak∗ limit of probability measures $\mu_T$ concentrated on $\mathcal{O}$.

**Proof of 4 ⇒ 1 (Katznelson, 1973).**

By hypothesis $\mu_T(f) \to \eta(f)$ for every continuous $f$ on $b$V, where the $\mu_T$ are concentrated on $\mathcal{O}$. If $f$ vanishes on the closure of $\mathcal{O}$ in $b$V then so do all $\mu_T(|f|)$ and hence also $\eta(|f|)$, which forces $f$ to vanish everywhere. So $\mathcal{O}$ is dense in $b$V. ∎

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One might wonder if 2 is equivalent to the *a priori* weaker:

2' $\mathcal{O}$ has dense image in any compact quotient group of V.

### Counterexample showing that 2' $\not\Rightarrow$ 2 (F. Jordan)

Take $\mathrm{V} = \mathbf{R}$ and $\mathcal{O} = \mathbf{Z} \cup 2\pi\mathbf{Z}$. Then $\mathcal{O}$ has dense image in every compact quotient $\mathbf{R}/a\mathbf{Z}$. Meanwhile, the morphism $\alpha : \mathbf{R} \to \mathbf{T}^2$ defined by $\alpha(v) = (e^{iv}, e^{2\pi i v})$ gives $\overline{\alpha(\mathcal{O})} = \mathbf{T} \times \{1\} \cup \{1\} \times \mathbf{T}$, which does not contain $\overline{\alpha(\mathrm{V})} = \mathbf{T}^2$.

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**Proof.**

This characteristic function is $\hat{\eta}$. So 2 says that

$$\mu_T(f) \to \eta(f) \qquad (*)$$

for *every continuous character* $f(\omega) = \omega(u)$ of $bV$. Whereas 1 says that $(*)$ holds for *every continuous function* $f$ on $bV$. But linear combinations of continuous characters are uniformly dense in the continuous functions on $bV$ (Stone-Weierstrass). So 1 and 2 imply each other. □

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1. The $\mu_T$ converge to Haar measure $\eta$ in the weak$*$ topology;
2. The Fourier transforms $\hat{\mu}_T(u) = \int \omega(u)\, d\mu_T(\omega)$ converge pointwise to the characteristic function of $\{0\} \subset V^*$.

## Proof.

This characteristic function is $\hat{\eta}$. So 2 says that

$$\mu_T(f) \to \eta(f) \qquad (*)$$

for *every continuous character* $f(\omega) = \omega(u)$ of $bV$. Whereas 1 says that $(*)$ holds for *every continuous function* $f$ on $bV$. But linear combinations of continuous characters are uniformly dense in the continuous functions on $bV$ (Stone-Weierstrass). So 1 and 2 imply each other. □

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**Lemma 3** (Van der Corput, 1921)

Suppose that $F : [a, b] \to \mathbf{R}$ is differentiable, its derivative $F'$ is monotone, and $|F'| \geqslant 1$ on $(a, b)$. Then $|\int_a^b e^{iF(t)}\, dt| \leqslant 3$.

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To this end, let

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denote the exponent in $(*)$. We are going to show that Lemma 3 applies to almost every $F_{kk'}$. In fact, it is well known that $\mathfrak{a}$ acts diagonalizably (over $\mathbf{R}$) on V. Thus, letting $E_\nu$ be the projector of V onto the weight $\nu$ eigenspace of $\mathfrak{a}$, we can write

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Indeed, suppose otherwise. Then, writing any $g \in G$ in the form $kak'$ (KAK decomposition, where $A = \exp(\mathfrak{a})$), we would have

$$\langle u, gv \rangle = \sum_{\nu \in \mathfrak{a}^*} \underbrace{\langle u, kE_\nu k' v \rangle}_{f_\nu(k,k')} e^{\langle \nu, \log(a) \rangle} = \underbrace{\langle u, kE_0 k' v \rangle}_{f_0(k,k')}.$$

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So we may pick a $\nu_0 \neq 0$ such that $f_{\nu_0}$ is not $\equiv 0$. Conjugating if necessary, we can assume that $\nu_0 \in C$, and choose it there so as to maximize $\langle \nu_0, H \rangle$. Then our exponent writes:

$$F_{kk'}(t) = e^{\langle \nu_0, H \rangle t} \left\{ f_{\nu_0}(k, k') + \varepsilon_{kk'}(t) \right\}$$

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# Proof of the Theorem

So Lemma 3 applies and gives

$$\left| \int_{T_0}^{T} e^{iF_{kk'}(t)}\, dt \right| \leqslant 3 \qquad \forall\, T.$$

Therefore we have $\lim_{T\to\infty} \frac{1}{T} \int_0^T e^{iF_{kk'}(t)}\, dt = 0$ for almost all $(k, k')$, whence the conclusion $(*)$ by dominated convergence. □

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# Outlook

For perspective, our result should be compared to

**Theorem** (Z., 1993)

Let $\mathcal{O}$ be the image of any polynomial map $\mathbf{R}^d \to V$ (V: finite-dimensional vector space). Then $\mathcal{O}$ has the same closure in $bV$ as its affine hull $\widehat{\mathcal{O}}$.

**Corollary**

Let G be a *nilpotent* Lie group and V a finite-dimensional G-module *of unipotent type*. Then any G-orbit $\mathcal{O}$ in V has the same closure in $bV$ as its affine hull $\widehat{\mathcal{O}}$.

**Remark.** The Corollary fails for V not of unipotent type, as one sees by observing that the orbits of $\mathbf{R}$ acting on $\mathbf{R}^2$ by $\exp\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$ (i.e., hyperbolas) already have non-dense images in $\mathbf{R}^2/\mathbf{Z}^2$.

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