Frobenius reciprocity: symplectic, prequantum, diffeological*

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Abstract: Ratiu–Z.† established "Frobenius reciprocity" as a bijection t between certain symplectically reduced spaces (which need not be manifolds), and conjectured:

- t is a diffeomorphism, relative to the subquotient diffeologies of these spaces;
- t respects the reduced diffeological 2-forms they may (or might not) carry.

We ***prove this***, and give ***new sufficient conditions*** for the reduced forms to exist.

* `arXiv:2403.3927`, joint with Gabriele Barbieri and Jordan Watts.

† `arXiv:2007.9434`, building on ideas of Guillemin–Sternberg (1983).

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Let (X, ω, Φ) be a Hamiltonian G-space (G: Lie group, Φ: equivariant moment map). The *reduced space*

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Definition

We say that $X/\!\!/G$ ***carries a reduced 2-form*** if there is a (diffeological) 2-form $\omega_{X/\!\!/G}$ such that $j^*\omega = \pi^*\omega_{X/\!\!/G}$, where

$$\begin{array}{ccc} \Phi^{-1}(0) & \xrightarrow{\;j\;} & X \\ {\scriptstyle\pi}\big\downarrow & & \\ X/\!\!/G. & & \end{array}$$

Note: we will see that if $\omega_{X/\!\!/G}$ exists, then it is unique (and closed).

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Example: $\mathrm{Hom}_G(X_1, X_2)$ (Guillemin-Sternberg 1982)

This is

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where (X_i, ω_i, Φ_i) are Hamiltonian G-spaces and $X^- := (X, -\omega, -\Phi)$. So the product here has diagonal G-action, 2-form $\omega_2 - \omega_1$, and moment map $\Phi(x_1, x_2) = \Phi_2(x_2) - \Phi_1(x_1)$.

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(D1) *Covering.* All constant maps $\mathbf{R}^n \to X$ are in $\mathcal{P}$, for all n.

(D2) *Locality.* Let $V \xrightarrow{P} X$ be a map with $V \in \tau_n$. If every point of V has an open neighborhood U such that $P|_U \in \mathcal{P}$, then $P \in \mathcal{P}$.

(D3) *Smooth compatibility.* Let $U \xrightarrow{\psi} V \xrightarrow{P} X$ be maps with $(U, V) \in \tau_m \times \tau_n$. If $P \in \mathcal{P}$ and $\psi \in C^\infty(U, V)$, then $P \circ \psi \in \mathcal{P}$.

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A map $(X, \mathcal{P}) \xrightarrow{F} (Y, \mathcal{Q})$ of diffeological spaces (: sets with diffeologies) is called ***smooth*** if $P \in \mathcal{P}$ implies $F \circ P \in \mathcal{Q}$.

If $(X, \mathcal{P}) \xrightarrow{\mathrm{id}} (X, \mathcal{Q})$ is smooth, i.e. $\mathcal{P} \subset \mathcal{Q}$, we call $\mathcal{P}$ ***finer*** and $\mathcal{Q}$ ***coarser***.

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- Let Y be a diffeological space and $i : X \to Y$ an injection. Then X has a coarsest diffeology making i smooth, the ***subset diffeology***. Its plots are the maps $P : U \to X$ such that $i \circ P$ is a plot of Y.

  Universal property: A map F *to* X is smooth iff $i \circ F$ is smooth.

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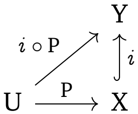

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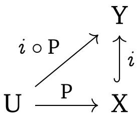

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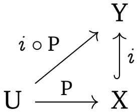

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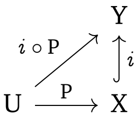

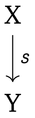

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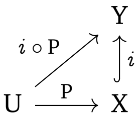

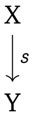

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$$\begin{array}{ccc} & & Y \\ & \overset{i\circ P}{\nearrow} & \uparrow i \\ U & \xrightarrow{P} & X \end{array} \qquad\qquad \begin{array}{ccc} & & X \\ & & \downarrow s \\ U & \xrightarrow{Q} & Y \end{array}$$

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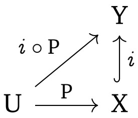

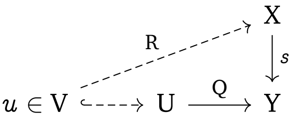

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Interlude: a historical question.

I have seen “strict” defined for

1. {topological groups, continuous morphisms}: Bourbaki 1960,
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Question

Is 4. really nowhere to be found before 1985?

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Definitions (Diffeological Cartan–de Rham calculus)

Let X and Y be diffeological spaces.

- A *k-form* α on X is a functional which sends each plot $P: V \to X$ to an ordinary k-form on V, denoted $P^*\alpha$.

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Theorem (Souriau's criterion, 1985)

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All diffeological notions used in §1 have now been defined.

Existence of $\omega_{X/\!\!/G}$: Prior State of the Art

- If the G-action on the level $C = \Phi^{-1}(0)$ is *locally free and proper*, it has long been known that $X/\!\!/G$ is an (effective) orbifold with an 'orbifold 2-form' (proof in Cushman–Bates 1997).

Briefly, §§4–6 will improve on this by showing: it suffices to assume *locally free* **or** *proper*, **or** *strict*.

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Let a diffeological group G act on a diffeological space X. We consider the map

$$\theta : G \times X \to X \times X, \qquad \theta(g, x) = (x, g(x)).$$

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The G-action is ***strict*** if θ is a strict map (§2).

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- A *free* action (θ injective) is strict iff it is *principal* (: θ induction). *Example:* any free action of a Lie G on a manifold (Iglesias 1985).
- A *transitive* action (θ surjective) is strict iff θ is a subduction. *Example:* any transitive action of a Lie G on a manifold.
- Non-free, non-transitive can easily be non-strict: e.g. $SO(2) \circlearrowright \mathbb{R}^2$.

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Let a diffeological group G act on a diffeological space X. We consider the map

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In the setting of §1, *suppose that the* G*-action on* $C = \Phi^{-1}(0)$ *is* ***strict***. *Then* $X /\!\!/ G$ *carries a reduced 2-form.*

Comments on the proof. Our above "smooth division"

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is just what's needed for a straightforward application of Souriau's criterion ♢ (using elementary properties of moment maps). Subtler results (§5, §6) tend to use ♢ in tandem with e.g. Baire category. □

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Example: $\mathrm{Ind}_H^G Y$

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$$\mathrm{Ind}_H^G Y := (T^*G \times Y)/\!\!/H = \psi^{-1}(0)/H \qquad \clubsuit$$

where: G is a Lie group, H is an *arbitrary subgroup* (hence canonically also a Lie group: Bourbaki 1972), (Y, ω_Y, Ψ) is a Hamiltonian H-space, and $L := T^*G \times Y$ is the Hamiltonian $G \times H$-space with action $(g,h)(p,y) = (gph^{-1}, h(y))$ and moment map $\phi \times \psi : L \to \mathfrak{g}^* \times \mathfrak{h}^*$,

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Instructive special case: $\mathrm{Ind}_H^G\{0\}$

This is $(T^*G)/\!\!/H$, the reduction of T^*G by the 'right' action of H.

- When H is closed, it is well known (Kummer–Marsden–Satzer) that $(T^*G)/\!\!/H = T^*(G/H)$ with its canonical 2-form and G-action.
- When H is not closed, Iglesias-Zemmour (2010) gave meaning to the right-hand side by defining, for any diffeological space X, a "cotangent space" $T^*(X)$ with a canonical 2-form d Liouv and Hamiltonian action of Diff(X).

We can ask, then, if the equality survives. It does at least for *dense* H:

Theorem 2

Let G *be a Lie group,* H *any dense subgroup. Then* $(T^*G)/\!\!/H = T^*(G/H)$ *as diffeological, parasymplectic Hamiltonian* G*-spaces.*

Example: G the 2-torus, H an irrational winding, $G/H = T_\alpha$.

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Instructive special case: $\mathrm{Ind}_H^G\{0\}$

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§7. Frobenius reciprocity

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Let G be a Lie group, H a closed subgroup, X a Hamiltonian G-space, Y a Hamiltonian H-space. Recall Hom_G and Ind_H^G, and define Res_H^G.

Theorem 6

There is a (diffeological) diffeomorphism

$$t : \mathrm{Hom}_G(X, \mathrm{Ind}_H^G Y) \to \mathrm{Hom}_H(\mathrm{Res}_H^G X, Y).$$

Moreover, if one side carries a reduced 2-form, then so does the other, and t maps one form to the other.

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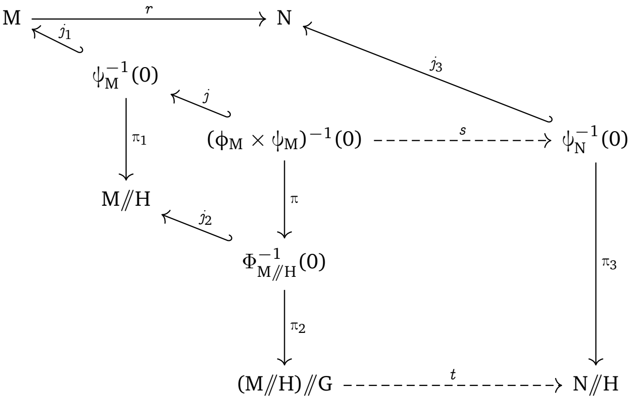

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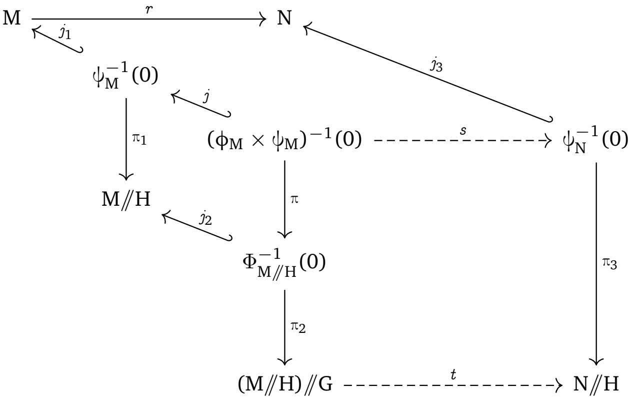

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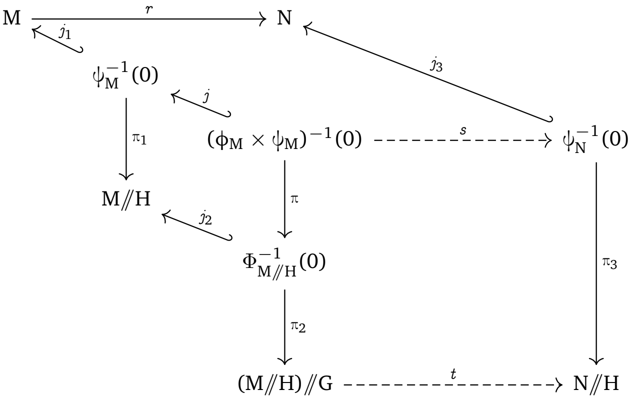

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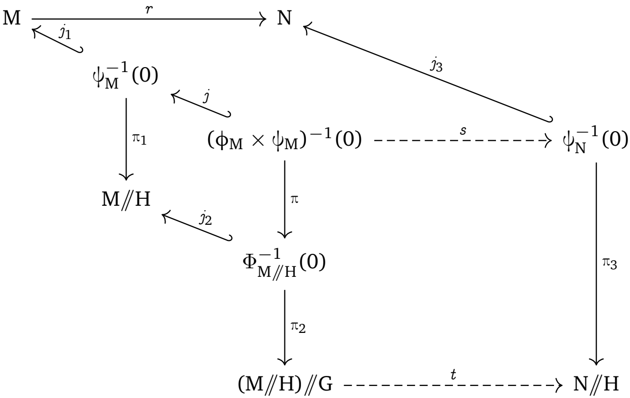

where the j's and π's are inclusions and projections as in §1. One checks that r sends $(\phi_M \times \psi_M)^{-1}(0)$ to $\psi_N^{-1}(0)$, so there is a map s as indicated; and s sends $G \times H$-orbits to H-orbits, so there is a map t. Likewise one checks that the right inverse $r' : N \to M$ defined by

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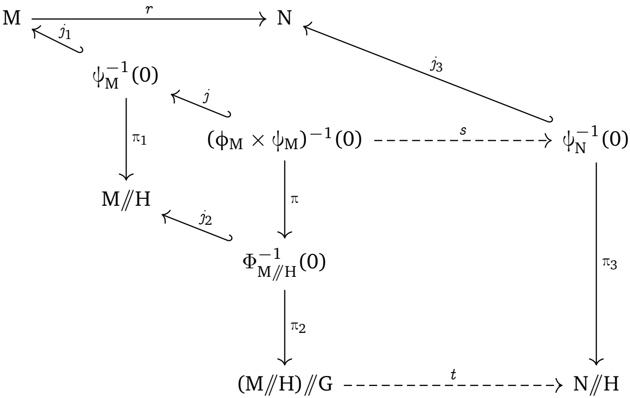

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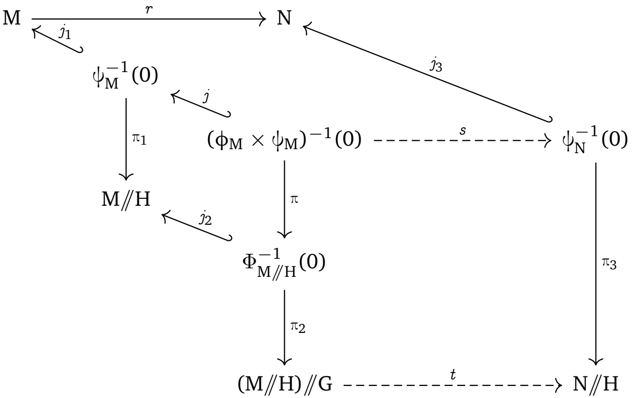

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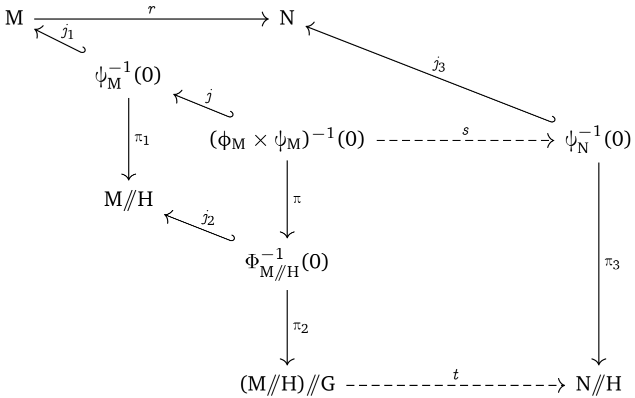

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Finally, assume merely that *one* reduced form exists, $\omega_{(\mathrm{M}/\!\!/\mathrm{H})/\!\!/\mathrm{G}}$ or $\omega_{\mathrm{N}/\!\!/\mathrm{H}}$. Then we can *define* the other by $\omega_{(\mathrm{M}/\!\!/\mathrm{H})/\!\!/\mathrm{G}} = t^*\omega_{\mathrm{N}/\!\!/\mathrm{H}}$; and an easy chase using again ($*$) shows that it is indeed a reduced form. □

Final remark: Everything we have said can be adapted to work also in the category {***prequantum G-spaces***}, which more closely mirrors the motivating category {unitary representations}. For details, see `arXiv:2007.9434` and `arXiv:2403.3927`.

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