Diffeological Reduced Spaces and Frobenius Reciprocity*

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Gone Fishing at Northwestern University
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Abstract: Ratiu-Z.† established "Frobenius reciprocity" as a bijection t between certain symplectically reduced spaces (which need not be manifolds), and conjectured:

- t is a diffeomorphism, relative to the subquotient diffeologies of these spaces;
- t respects the reduced diffeological 2-forms they may (or might not) carry.

We prove this, and give new sufficient conditions for the reduced forms to exist.

*arXiv:2403.3927, joint with Gabriele Barbieri and Jordan Watts.

†arXiv:2007.9434, building on ideas of Guillemin-Sternberg (1983).

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Let (X, ω, Φ) be a Hamiltonian G-space (G: Lie group, Φ: equivariant moment map). The *reduced space*

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need not be a manifold; but it has a natural ("subquotient") *diffeology*. It may or might not carry a reduced 2-form:

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Definition

We say that $X/\!\!/G$ ***carries a reduced 2-form*** if there is a (diffeological) 2-form $\omega_{X/\!\!/G}$ such that $j^*\omega = \pi^*\omega_{X/\!\!/G}$, where

$$\begin{array}{ccc} \Phi^{-1}(0) & \xrightarrow{j} & X \\ \downarrow{\scriptstyle\pi} & & \\ X/\!\!/G. & & \end{array}$$

Note: we will see that if $\omega_{X/\!\!/G}$ exists, then it is unique (and closed).

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Important special case (which sounds more general):

Example: $\mathrm{Hom}_G(X_1, X_2)$ (Guillemin-Sternberg 1982)

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where (X_i, ω_i, Φ_i) are Hamiltonian G-spaces and $X^- := (X, -\omega, -\Phi)$. So the product here has diagonal G-action, 2-form $\omega_2 - \omega_1$, and moment map $\Phi(x_1, x_2) = \Phi_2(x_2) - \Phi_1(x_1)$.

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- More generally, Guillemin-Sternberg took for X_1 a coadjoint orbit $G(\mu)$, and noted that ♡ then boils down to the space $\Phi_2^{-1}(\mu)/G_\mu$ of Marsden-Weinstein: this is their famous "shifting trick".

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Define $\mathcal{P} := \bigcup_{n \in \mathbf{N},\, U \in \tau_n} C^\infty(U, X)$. This satisfies:

(D1) *Covering.* All constant maps $\mathbf{R}^n \to X$ are in $\mathcal{P}$, for all n.

(D2) *Locality.* Let $V \xrightarrow{P} X$ be a map with $V \in \tau_n$. If every point of V has an open neighborhood U such that $P|_U \in \mathcal{P}$, then $P \in \mathcal{P}$.

(D3) *Smooth compatibility.* Let $U \xrightarrow{\varphi} V \xrightarrow{P} X$ be maps with $(U, V) \in \tau_m \times \tau_n$. If $P \in \mathcal{P}$ and $\varphi \in C^\infty(U, V)$, then $P \circ \varphi \in \mathcal{P}$.

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Let X be a set. A ***diffeology*** on X is a subset $\mathcal{P}$ of $\bigcup_{n\in\mathbf{N},\, U\in\tau_n} \mathrm{Maps}(U,X)$ satisfying (D1–D3). We call its members with domain $U \in \tau_n$, ***n-plots***.

A map $(X,\mathcal{P}) \xrightarrow{F} (Y,\mathcal{Q})$ of diffeological spaces (: sets with diffeologies) is called *smooth* if $P \in \mathcal{P}$ implies $F \circ P \in \mathcal{Q}$.

If $(X,\mathcal{P}) \xrightarrow{\mathrm{id}} (X,\mathcal{Q})$ is smooth, i.e. $\mathcal{P} \subset \mathcal{Q}$, we call $\mathcal{P}$ *finer* and $\mathcal{Q}$ *coarser*.

E.g.: {locally constant maps} $=: \mathcal{P}_{\text{discrete}} \subset \mathcal{P} \subset \mathcal{P}_{\text{coarse}} :=$ {all maps}.

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- Let Y be a diffeological space and $i : \mathrm{X} \to \mathrm{Y}$ an injection. Then X has a coarsest diffeology making i smooth, the ***subset diffeology***. Its plots are the maps $\mathrm{P} : \mathrm{U} \to \mathrm{X}$ such that $i \circ \mathrm{P}$ is a plot of Y.

  Universal property: A map F *to* X is smooth iff $i \circ \mathrm{F}$ is smooth.

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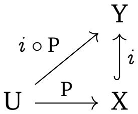

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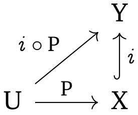

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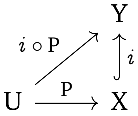

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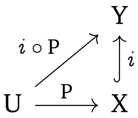

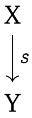

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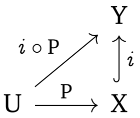

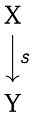

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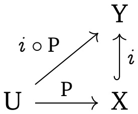

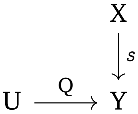

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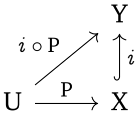

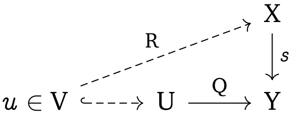

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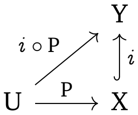

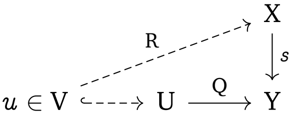

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- Its *exterior derivative* $d\alpha$ is the $(k+1)$-form defined for all plots P of Y by $P^* d\alpha = dP^*\alpha$, with ordinary d on the right-hand side.

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Let us call *ordinary* the k-forms on Euclidean open sets and operations on them (pull-back, exterior derivative).

Definitions (Diffeological Cartan-de Rham calculus)

Let X and Y be diffeological spaces.

- A ***k-form*** α on Y is a functional which sends each plot $P : V \to Y$ to an ordinary k-form on V, denoted $P^*\alpha$. As compatibility, we require: if $\psi \in C^\infty(U, V)$ (so $P \circ \psi$ is another plot), then

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Comments on the proof. Necessity is clear: if $\alpha = s^*\beta$, we have

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by definition of s^*; ♢ follows. Proving the rest takes about 2 pages. □

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There is a basic criterion for when a k-form descends to a quotient:

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All diffeological notions used in §1 have now been defined.

Existence of $\omega_{X/\!\!/G}$: Prior State of the Art

- If the G-action on the level $C = \Phi^{-1}(0)$ is *locally free and proper*, it has long been known that $X/\!\!/G$ is an (effective) orbifold with an 'orbifold 2-form' (proof in Cushman-Bates 1997). Now when orbifolds are regarded as diffeological spaces, 'orbifold forms' define diffeological forms and conversely (Karshon-Watts 2016).

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Briefly, §§4–6 will improve on this by showing: it suffices to assume *locally free* or *proper*, or *strict*.

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- Note: locally free means that the infinitesimal stabilizer $\mathfrak{g}_z$ is zero for all $z \in C$. As $\mathrm{Im}(D\Phi(z)) = \mathrm{annihilator}(\mathfrak{g}_z)$, it follows that 0 is a regular value, so C is a manifold.

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§4. Strict actions

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Let a diffeological group G act on a diffeological space X. We consider the map

$$\theta : G \times X \to X \times X, \qquad \theta(g,x) = (x, g(x)).$$

Definition

The G-action is ***strict*** if θ is a strict map (§2).

Theorem 1

In the setting of §1, suppose that the G-action on $C = \Phi^{-1}(0)$ is ***strict****. Then $X /\!\!/ G$ carries a reduced 2-form.*

Comments on the proof. Strictness is "just what's needed" for a straightforward application of Souriau's criterion ♢. Subtler results (§5, §6) tend to use ♢ in tandem with e.g. Baire category. □

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Example: $\mathrm{Ind}_H^G Y$

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Instructive special case: $\mathrm{Ind}_H^G\{0\}$

This is $(T^*G)/\!\!/H$, the reduction of T^*G by the 'right' action of H.

- When H is closed, it is well known (Kummer-Marsden-Satzer) that $(T^*G)/\!\!/H = T^*(G/H)$ with its canonical 2-form and G-action.
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We can ask, then, if the equality survives. It does at least for *dense* H:

Theorem 2

Let G *be a Lie group,* H *any dense subgroup. Then* $(T^*G)/\!\!/H = T^*(G/H)$ *as diffeological, parasymplectic Hamiltonian* G*-spaces.*

Example: G the 2-torus, H an irrational winding, $G/H = \mathbf{T}_\alpha$.

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In the setting of §1, *suppose the* G*-action on* $C = \Phi^{-1}(0)$ *is* ***locally free*** *(i.e. all* $x \in C$ *have infinitesimal stabilizer* $\mathfrak{g}_x = \{0\}$*) and* G *is connected. Then* $X /\!\!/ G$ *carries a reduced 2-form.*

Comments on the proof. Under the hypotheses, standard properties of the moment map:

$$\operatorname{Ker}(D\Phi(x)) = \mathfrak{g}(x)^{\omega}, \qquad \operatorname{Im}(D\Phi(x)) = \operatorname{annihilator}(\mathfrak{g}_x)$$

readily imply that 1) 0 is a regular value of Φ, 2) C is a submanifold, 3) in C the G-orbits are the leaves of a *foliation* $\mathcal{F}$, 4) $\omega_{|C} := j^*\omega$ is *basic* for $\mathcal{F}$, i.e., G-invariant with $\mathfrak{g}(x) \subset \operatorname{Ker}(\omega_{|C})$. A theorem on foliations by Hector *et al.* (2011) then implies the result. □

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§7. Frobenius reciprocity

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Let G be a Lie group, H a closed subgroup, X a Hamiltonian G-space, Y a Hamiltonian H-space. Recall Hom_G and Ind_H^G, and define Res_H^G.

Theorem 5

There is a (diffeological) diffeomorphism

$$t : \mathrm{Hom}_G(X, \mathrm{Ind}_H^G Y) \to \mathrm{Hom}_H(\mathrm{Res}_H^G X, Y).$$

Moreover, if one side carries a reduced 2-form, then so does the other, and t maps one form to the other.

Sketch of proof. The sides are respectively $(M /\!\!/ H) /\!\!/ G$ and $N /\!\!/ H$, where

$$M = X^- \times T^*G \times Y, \qquad \text{resp.} \qquad N = X^- \times Y$$

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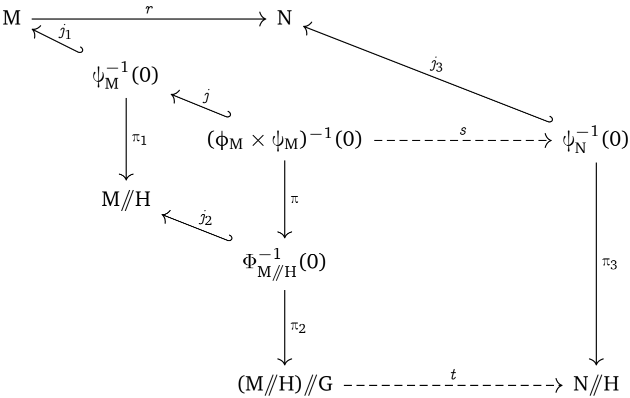

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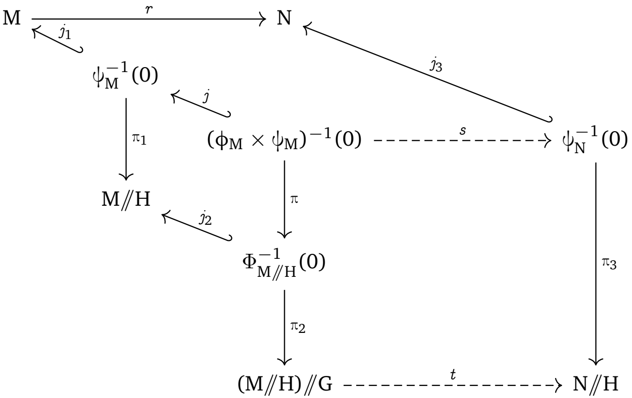

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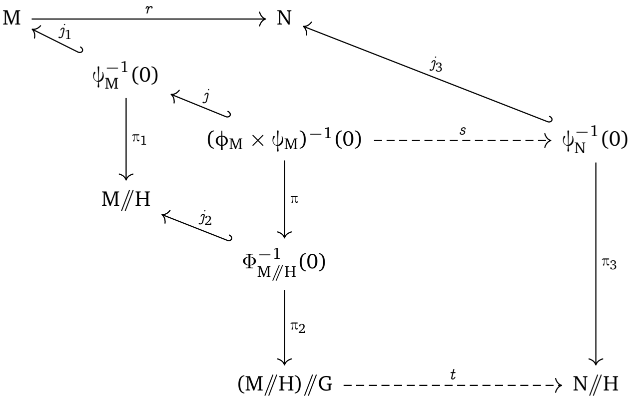

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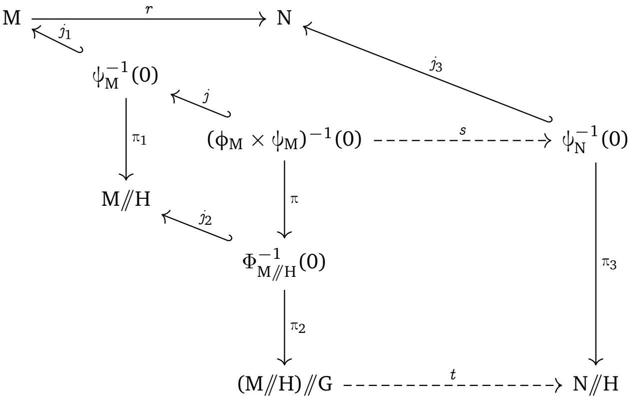

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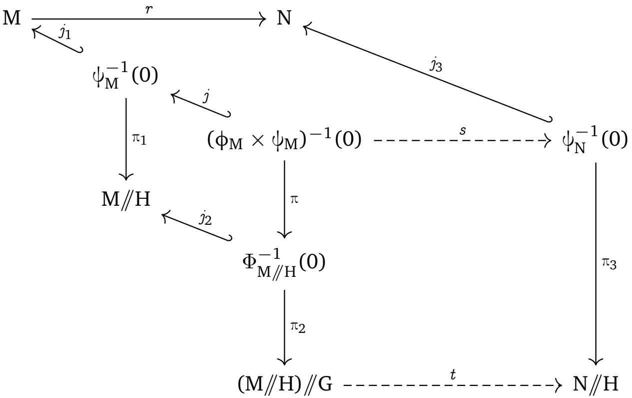

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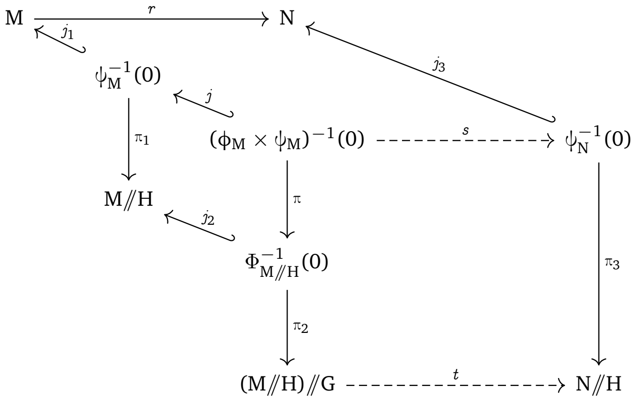

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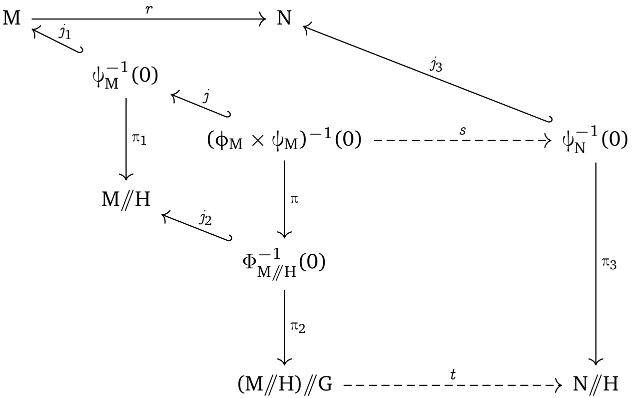

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Next, assume that both sides carry reduced 2-forms, $\omega_{(\mathrm{M}/\!\!/\mathrm{H})/\!\!/\mathrm{G}}$ and $\omega_{\mathrm{N}/\!\!/\mathrm{H}}$. We must prove $\omega_{(\mathrm{M}/\!\!/\mathrm{H})/\!\!/\mathrm{G}} = t^*\omega_{\mathrm{N}/\!\!/\mathrm{H}}$. By chasing the diagram, one checks that this is equivalent to, not quite $\omega_\mathrm{M} = r^*\omega_\mathrm{N}$ but

$$j^*j_1^*\omega_\mathrm{M} = j^*j_1^*r^*\omega_\mathrm{N}, \qquad (*)$$

an equality of 2-forms on $(\phi_\mathrm{M} \times \psi_\mathrm{M})^{-1}(0)$ (usually not a manifold). Now $(*)$ means that its sides coincide after pull-back by any plot P of that subset, i.e., by any *smooth map* $\mathrm{P} : \mathrm{U} \to \mathrm{M}$ *taking values in* $(\phi_\mathrm{M} \times \psi_\mathrm{M})^{-1}(0)$. This is true, and can be checked in about 10 lines.

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