# Relativity without light: A new proof of Ignatowski's theorem*

Jean-Philippe Anker (Université d'Orléans)
François Ziegler (Georgia Southern)

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**Abstract:** Ignatowski (1910) showed that assumptions about light are not needed to obtain Lorentzian kinematics as one of only few possibilities. We give a much simplified proof of his result as formulated by Gorini (1971) for $n+1$-dimensional space-time.

*http://arxiv.org/abs/2007.09301

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- His argument is a classic example of how *group theory* is powerful to describe Nature.
- New proof illustrates what can be done with *Lie theory* and a little *representation theory*.

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When $\sigma = 0$ this still makes sense and is called the *Galilei* group.

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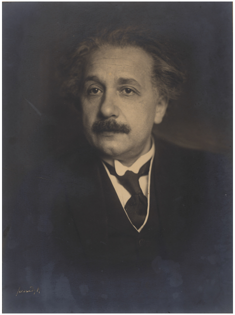

Einstein (1905)

Poincaré (1905)

Minkowski (1908)

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Als Einstein seinerzeit das Relativitätsprinzip einführte, nahm er parallel mit demselben an, daß die Lichtgeschwindigkeit $c$ eine universelle Konstante sei, d.h. für alle Koordinatensysteme denselben Wert behalte. Auch Minkowski ging bei seinen Untersuchungen von der Invariante $r^2 - c^2t^2$ aus, obwohl nach seinem Vortrage „Raum und Zeit"[1]) zu urteilen, er dem $c$ mehr die Bedeutung einer universellen Raum—Zeit-Konstante beilegte, als diejenige der Lichtgeschwindigkeit.

Nun habe ich mir die Frage gestellt, zu welchen Beziehungen bezw. Transformationsgleichungen man kommt, wenn man nur das Relativitätsprinzip an die Spitze der Untersuchung stellt und ob überhaupt die Lorentzschen Transformationsgleichungen die einzigen sind, die dem Relativitätsprinzip genügen.

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## Photon and graviton mass limits

Alfred Scharff Goldhaber

*C. N. Yang Institute for Theoretical Physics, SUNY Stony Brook, New York 11794-3840, USA and Theoretical Division (MS B285), Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*

Michael Martin Nieto

*Theoretical Division (MS B285), Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*

(Published 23 March 2010)

Efforts to place limits on deviations from canonical formulations of electromagnetism and gravity have probed length scales increasing dramatically over time. Historically, these studies have passed through three stages: (1) testing the power in the inverse-square laws of Newton and Coulomb, (2) seeking a nonzero value for the rest mass of photon or graviton, and (3) considering more degrees of

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$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{g} \begin{pmatrix} x \\ t \end{pmatrix}. \quad (1)$$

As (1) maps the line $x = 0$ to the line $x' = \frac{b}{d} t'$, he calls $v := \frac{b}{d}$ the *velocity* of the transformation. He requires that the velocity $\frac{-b}{a}$ of the inverse transformation $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ be opposite to $v$, whence

$$a = d. \quad (2)$$

Now in order that a product $\begin{pmatrix} a & b \\ c & a \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & a' \end{pmatrix} = \begin{pmatrix} aa'+bc' & * \\ * & aa'+b'c \end{pmatrix}$ still satisfy (2) we must have $bc' = b'c$, i.e. there is a constant $\sigma$ such that

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So our transformations have the form $g = \begin{pmatrix} a & b \\ \sigma b & a \end{pmatrix}$ with $b = av$. Next Ignatowski argues that G should contain the reflection $\mathrm{R} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and hence also the product $\mathrm{R}g\mathrm{R}g = \begin{pmatrix} a^2-\sigma b^2 & 0 \\ 0 & a^2-\sigma b^2 \end{pmatrix}$, but no scalar matrices other than the identity (physics is not dilation-invariant). Hence we should have $1 = a^2 - \sigma b^2 \underset{\underset{b=av}{\uparrow}}{=} a^2(1-\sigma v^2)$ and therefore

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Summing up, we have obtained

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**Grundlagen zu einer Theorie der Lorentztransformationen.**

Von Emil Hahn in Straßburg i. Els.

**Einleitung.**

Die Welt des von Einstein[1]) und Minkowski[2]) eingeführten Raum-Zeitgebildes hat auf den ersten Blick etwas Befremdendes und, wie Minkowski[3]) selbst bemerkt hat, Mißfälliges. Dieses Befremden schwindet sehr, wenn man den Voraussetzungen nachgeht, auf denen das Ganze beruht. Der entscheidende Schritt in dieser Richtung wurde von Ignatowsky[4]) getan, der den Beweis zu führen suchte, daß das An-

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## Linear Kinematical Groups

VITTORIO GORINI★

Institut für Theoretische Physik (I) der Universität Marburg, Germany

Received November 12, 1970

**Abstract.** We prove a theorem which states that in an $(n+1)$-dimensional space-time $(n \geqq 3)$ the only linear kinematical groups which are compatible with the isotropy of space are the Lorentz and Galilei groups. The special cases $n=1$ and $n=2$ are also briefly discussed.

# Gorini's theorem (1971)



## Theorem (Gorini 1971)

*Suppose* $n \geqslant 2$ *and let* $G$ *be a subgroup of* $GL_{n+1}(\mathbf{R})$ *such that*

$$G \cap \begin{pmatrix} GL_n(\mathbf{R}) & 0 \\ 0 & \mathbf{R}^\times \end{pmatrix} = \begin{pmatrix} O_n(\mathbf{R}) & 0 \\ 0 & \pm 1 \end{pmatrix}. \tag{6}$$

*Write* $K$ *for the right-hand side of* (6). *Then either* $G = K$ *or there is a number* $\sigma \in \mathbf{R} \cup \{\infty\}$ *such that*

$$G = K \exp(\mathfrak{p}_\sigma), \tag{7}$$

*where* $\mathfrak{p}_\sigma = \left\{ \begin{pmatrix} 0 & b \\ \sigma \overline{b} & 0 \end{pmatrix} : b \in \mathbf{R}^n \right\}$ *and* $\mathfrak{p}_\infty = \left\{ \begin{pmatrix} 0 & 0 \\ \overline{c} & 0 \end{pmatrix} : c \in \mathbf{R}^n \right\}$.

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In other words, G under the hypotheses must be isomorphic to one of these 5 possibilities — all of them well-known and named:

# Gorini's theorem (1971)



1. if $\sigma > 0$, the Lorentz group $O_{n,1}(\mathbf{R})$ (Poincaré, 1906);
2. if $\sigma = 0$, the homogeneous Galilei group (Frank, 1908);
3. if $\sigma < 0$, the orthogonal group $O_{n+1}(\mathbf{R})$ (Jordan, 1870);
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Gorini's result is remarkable, but didn't catch on — perhaps because his *proof* was neither concise nor enlightening.

# Gorini's theorem (1971)



By Lemma 1 (see (4.2)), we have

$$\frac{d(v)+vc(v)\cos\varepsilon}{d(v)+vc(v)}>0\,,\qquad \varepsilon\in[0,2\pi)\,, \tag{4.25}$$

and this gives the inequality

$$|vc(v)/d(v)|<1\,,\qquad v\in\mathscr{V}\,. \tag{4.26}$$

Defining

$$\cos\gamma(\varepsilon,v)=\frac{de(1-\cos\varepsilon)}{[d^2e^2(1-\cos\varepsilon)^2+a^2(d+vc)^2\sin^2\varepsilon]^{1/2}}$$

and

$$\sin\gamma(\varepsilon,v)=\frac{a(d+vc)\sin\varepsilon}{[d^2e^2(1-\cos\varepsilon)^2+a^2(d+vc)^2\sin^2\varepsilon]^{1/2}}\,,$$

with $\sin\gamma(0,v)=\sin\gamma(+0,v)=1$ and $\cos\gamma(0,v)=\cos\gamma(+0,v)=0$, we calculate

$$X(\varepsilon,v)=C^{(1,2)}(\pi-\gamma(\varepsilon,v))\,L(\varepsilon,v)\,C^{(1,2)}(-\gamma(\varepsilon,v))$$

$$=\begin{pmatrix} \dfrac{d+vc\cdot\cos\varepsilon}{d+vc} & \varphi_1(a,c,d,e,v,\varepsilon) & 0\ldots0 & \varphi_2(a,c,d,e,v,\varepsilon)\\ 0 & 1 & 0\ldots0 & 0\\ 0 & 0 & 1\ldots0 & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0\ldots1 & 0\\ \varphi_3(a,c,d,e,v,\varepsilon) & \varphi_4(a,c,d,e,v,\varepsilon) & 0\ldots0 & \dfrac{d+vc\cdot\cos\varepsilon}{d+vc} \end{pmatrix}, \tag{4.27}$$

where

$$\varphi_1(a,c,d,e,v,\varepsilon)$$
$$=\frac{\sin\varepsilon\{de^2(1-\cos\varepsilon)\,[d^2e^2(1-\cos\varepsilon)+a^2(d+vc)^2\cos\varepsilon]}{ae(d+vc)\,[d^2e^2(1-\cos\varepsilon)^2} \tag{4.27a}$$
$$\frac{-a^2(d+vc)[a^2(d+vc)^2\sin^2\varepsilon-de^2(1-\cos\varepsilon)(d\cdot\cos\varepsilon+vc)]\}}{+a^2(d+vc)^2\sin^2\varepsilon]}\,,$$

$$\varphi_2(a,c,d,e,v,\varepsilon)=\frac{-v[d^2e^2(1-\cos\varepsilon)^2+a^2(d+vc)^2\sin^2\varepsilon]^{1/2}}{e(d+vc)}\,, \tag{4.27b}$$

$$\varphi_3(a,c,d,e,v,\varepsilon)=\frac{ce[2d(1-\cos\varepsilon)+vc\sin^2\varepsilon]}{(d+vc)[d^2e^2(1-\cos\varepsilon)^2+a^2(d+vc)^2\sin^2\varepsilon]^{1/2}} \tag{4.27c}$$



and

$$\varphi_4(a,c,d,e,v,\varepsilon)=\frac{c\sin\varepsilon(1-\cos\varepsilon)[e^2d-a^2(d+vc)]}{a(d+vc)[d^2e^2(1-\cos\varepsilon)^2+a^2(d+vc)^2\sin^2\varepsilon]^{1/2}}\,. \tag{4.27d}$$

One checks that $\Delta(X(\varepsilon,v))=1$, hence $X(\varepsilon,v)$ is an element of $\mathscr{K}$ and, by Lemma 3, $X(\varepsilon,v)\in\mathscr{N}$. Therefore,

$$X(\varepsilon,v)=N(w(\varepsilon,v))\,, \tag{4.28}$$

where, by (4.27b),

$$w(\varepsilon,v)=\frac{v[d^2e^2(1-\cos\varepsilon)^2+a^2(d+vc)^2\sin^2\varepsilon]^{1/2}}{e(d+vc\cos\varepsilon)}\,. \tag{4.29}$$

Eqs. (4.14) and (4.15) imply

$$\varphi_1(a,c,d,e,v,\varepsilon)=\varphi_4(a,c,d,e,v,\varepsilon)=0\,. \tag{4.30}$$

One checks that for (4.30) to be satisfied it is necessary and sufficient that

$$d(v)\,e^2(v)=a^2(v)\,[d(v)+v\,c(v)]\,. \tag{4.31}$$

Using (4.31) we get

$$d^2e^2(1-\cos\varepsilon)^2+a^2(d+vc)^2\sin^2\varepsilon=de^2[2d(1-\cos\varepsilon)+vc\sin^2\varepsilon]\,. \tag{4.32}$$

We have from (4.27) and (4.28), $\forall\,\varepsilon\in[0,2\pi)$,

$$e(w(\varepsilon,v))=f(w(\varepsilon,v))=1\,, \tag{4.33a}$$

$$a(w(\varepsilon,v))=d(w(\varepsilon,v))=\frac{d+vc\cos\varepsilon}{d+vc} \tag{4.33b}$$

and, using (4.32),

$$c(w(\varepsilon,v))=\frac{c\{d[2d(1-\cos\varepsilon)+vc\sin^2\varepsilon]\}^{1/2}}{d(d+vc)} \tag{4.33c}$$

and

$$w(\varepsilon,v)=\frac{v\{d[2d(1-\cos\varepsilon)+vc\sin^2\varepsilon]\}^{1/2}}{d+vc\cos\varepsilon}\,. \tag{4.33d}$$

From (4.33b, c, d) we get, $\forall\,\varepsilon\in[0,2\pi)$,

$$1+w^2(\varepsilon,v)\,\frac{c(v)}{vd(v)}>0\,, \tag{4.34}$$

$$a(w(\varepsilon,v))=\left(1+\frac{c(v)}{vd(v)}\,w^2(\varepsilon,v)\right)^{-1/2} \tag{4.35}$$

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# Our proof

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## Step 1: Lie group structure of G

G admits a canonical ("initial") Lie group structure having Lie algebra

$$\mathfrak{g} := \left\{ Z \in \mathfrak{gl}_{n+1}(\mathbf{R}) : e^{tZ} \in G \text{ for all } t \in \mathbf{R} \right\}. \tag{8}$$

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Because G contains K, (8) clearly contains the Lie algebra $\mathfrak{k}$ of K. And because $e^{tkZk^{-1}} = ke^{tZ}k^{-1}$, $\mathfrak{g}$ is also an *invariant* subspace of $\mathfrak{gl}_{n+1}(\mathbf{R})$ under the adjoint representation of $O_n(\mathbf{R}) \subset K$:

$$\mathrm{Ad}\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & b \\ \overline{c} & d \end{pmatrix} = \begin{pmatrix} RAR^{-1} & Rb \\ \overline{Rc} & d \end{pmatrix}, \quad R \in O_n(\mathbf{R}). \quad (9)$$

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Because G contains K, (8) clearly contains the Lie algebra $\mathfrak{k}$ of K. And because $e^{tkZk^{-1}} = ke^{tZ}k^{-1}$, $\mathfrak{g}$ is also an *invariant* subspace of $\mathfrak{gl}_{n+1}(\mathbf{R})$ under the adjoint representation of $O_n(\mathbf{R}) \subset K$:

$$\mathrm{Ad}\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & \boldsymbol{b} \\ \overline{\boldsymbol{c}} & d \end{pmatrix} = \begin{pmatrix} RAR^{-1} & R\boldsymbol{b} \\ \overline{R\boldsymbol{c}} & d \end{pmatrix}, \quad R \in O_n(\mathbf{R}). \tag{9}$$

# Our proof

## Step 2: Determination of $\mathfrak{g}$

Either $\mathfrak{g} = \mathfrak{k}$ or there is $\sigma \in \mathbf{R} \cup \{\infty\}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_\sigma$.

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$$\left[\begin{pmatrix} 0 & \boldsymbol{b} \\ \overline{\boldsymbol{c}} & 0 \end{pmatrix}, \left[\begin{pmatrix} 0 & \boldsymbol{b} \\ \overline{\boldsymbol{c}} & 0 \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right]\right] = \begin{pmatrix} * & 0 \\ 0 & 2\left\{\|\boldsymbol{b}\|^2\|\boldsymbol{c}\|^2 - (\overline{\boldsymbol{b}}\boldsymbol{c})^2\right\} \end{pmatrix}. \quad (10)$$

As this is contained in $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{k}]] \subset \mathfrak{g}$, the lower right entry must be 0: so the Cauchy-Schwarz bound is attained, i.e. $\boldsymbol{b}$ and $\boldsymbol{c}$ are collinear.

So each $Z \in \mathfrak{g} \cap M_3$ is in some $\mathfrak{p}_\sigma$. Moreover $\sigma$ must be the same for any two nonzero members $Z_1, Z_2$: else, one would readily find — by considering linear combinations of $Z_1$ and $\mathrm{Ad}(k)(Z_2)$ with $k \in K$ — that $\mathfrak{g} \cap M_3$ fills all $M_3$, hence $\mathfrak{g} = \mathfrak{k} \oplus M_3$, which by (10) isn't a Lie subalgebra. So Step 2 is complete. □

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## Step 3: Passage from $\mathfrak{g}$ to G

With $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_\sigma$ as in Step 2, we have $K \exp(\mathfrak{p}_\sigma) \subset G \subset N(\mathfrak{g})$, where

$$N(\mathfrak{g}) := \left\{ a \in GL_{n+1}(\mathbf{R}) : a\mathfrak{g}a^{-1} \subset \mathfrak{g} \right\}. \tag{11}$$

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*Proof.* The first inclusion holds by hypothesis and definition of $\mathfrak{g}$, the second because *any* subgroup G of any Lie group is contained in the normalizer (11) of its Lie algebra (since $e^{tgZg^{-1}} = ge^{tZ}g^{-1}$). □

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## Step 4: Determination of $\mathrm{N}(\mathfrak{g})$

$$\mathrm{N}(\mathfrak{k} \oplus \mathfrak{p}_\sigma) = \begin{cases} \mathbf{R}^\times \mathrm{K}\exp(\mathfrak{p}_\sigma) & \text{if } \sigma \in \mathbf{R}^\times, \\ \begin{pmatrix} \mathbf{R}^\times \mathbf{1} & 0 \\ 0 & \mathbf{R}^\times \end{pmatrix} \mathrm{K}\exp(\mathfrak{p}_\sigma) & \text{if } \sigma \in \{0, \infty\} \text{ or } \mathfrak{p}_\sigma = \{0\}. \end{cases} \tag{12}$$

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*Proof omitted.* (A simple computation using Schur's lemma.) □

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Steps 3 and 4, plus our hypothesis implying that $\begin{pmatrix} \lambda\mathbf{1} & 0 \\ 0 & \mu \end{pmatrix}$ is not in G unless it is in K (so $\lambda, \mu = \pm 1$), complete the theorem's proof. □

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More details at http://arxiv.org/abs/2007.09301.