Contact Imprimitivity

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Representation theory

(background)

Systems of imprimitivity

Let G be a locally compact group (e.g. Lie), V a unitary G-module.

Definition

- A *system of imprimitivity* for V is a G-invariant, commutative C^*-subalgebra $A \subset \mathrm{End}(V)$.
- Its *base* is its Gelfand spectrum $B = \{\text{nonzero *-homomorphisms } b : A \to \mathbf{C}\}$, with topology of pointwise convergence.

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Remark: The *Gelfand transform* $a \mapsto \hat{a}$, defined by $\hat{a}(b) = b(a)$, is an isomorphism $A \to C_0(B)$. Its inverse E is a *-representation of $C_0(B)$ in V such that

$$E(f \circ g_B^{-1}) = g_V E(f) g_V^{-1},$$

i.e. a "system of imprimitivity" in the original Mackey-Blattner sense.

The imprimitivity theorem

The point of this is:

Theorem (Frobenius, Mackey)

The following are equivalent:

- V *admits a transitive system of imprimitivity with base* $B = G/H$ ($H = G_b$ *say*);
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Explanation (case G/H admits a G-invariant measure):

$\Uparrow$: $\mathrm{Ind}_H^G W := \{L^2$ sections s of associated bundle $G \times_H W \to G/H\}$.
This indeed admits a system of imprimitivity, viz.: $E_{\mathrm{ind}}(f)s = fs$.

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Key application: The Mackey-Clifford *normal subgroup analysis*: When G has a closed normal subgroup N, classify unirreps of G in terms of unirreps of N and projective unirreps of subgroups of G/N.

Symplectic imprimitivity

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Let (X, σ, Φ) be a hamiltonian G-space (2-form σ, *equiv.* moment Φ).

Definition

- A *system of imprimitivity* for X is a G-invariant, commutative Lie subalgebra $\mathfrak{f} \subset C^\infty(X)$, such that the hamiltonian vector field $\mathrm{drag}\, f$ is complete for all $f \in \mathfrak{f}$.
- Its *base* is the image S of the "moment map" $\kappa: X \to \mathfrak{f}^*$, $(\kappa(x), f) = f(x)$. Each $f \in \mathfrak{f}$ descends to a function $\bar{f}$ on S.

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- The base, B, is a *G-subset* of $\mathfrak{f}^*$: $\langle g_{\mathrm{B}}(b), f \rangle = \langle b, f \circ g_{\mathrm{X}} \rangle$.
- The system, $\mathfrak{f}$, is called *transitive* if 1°) G acts transitively on B, 2°) $\pi : \mathrm{X} \to \mathrm{B}$ is C^∞ for the homogeneous space structure on B.

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Explanation: $\mathcal{F} :=$ ($\mathfrak{f}$ as an additive group) acts on X by $f_X = e^{\operatorname{drag} f}$ and π is *formally* a moment map for this action: $\operatorname{drag}\langle \pi(\cdot), f \rangle = \operatorname{drag} f$. Stabilizers G_b are *closed* so B's homogeneous structure is well-defined.

Symplectic imprimitivity theorem

Theorem (Z. [arXiv:1410.7950])

The following are equivalent:

- X *admits a transitive system of imprimitivity with base* $B = G/H$ ($H = G_b$ *say*);
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1. Endow $\mathrm{M} := \mathrm{T}^*\mathrm{G} \times \mathrm{Y}$ with the 2-form $\omega = d\theta + \tau$, $\theta = $ "$\langle p, dq\rangle$".
2. Let H act on M by $h(p, y) = (ph^{-1}, h(y))$.
3. This has moment map $\phi(p, y) = \Psi(y) - q^{-1}p_{|\mathfrak{h}}$ $(p \in \mathrm{T}^*_q\mathrm{G})$.
4. Define $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\,\mathrm{Y} = \phi^{-1}(0)/\mathrm{H}$ (Marsden-Weinstein subquotient).
5. The G-action $g(p, y) = (gp, y)$ with moment map $\varphi(p, y) = pq^{-1}$ passes to the quotient, whence the claimed G-space structure.

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$$\pi_{\mathrm{ind}} : \mathrm{Ind}_H^G Y \to G/H$$

arises by noting that the map $T^*G \times Y \to G/H$ sending $T^*_q G \times Y$ to qH is constant on H-orbits, hence passes to the (sub)quotient. Then one checks that

$$\mathfrak{f}_{\mathrm{ind}} := \pi^*_{\mathrm{ind}}(C^\infty(G/H))$$

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Explanation:

$\Downarrow$: Explicitly Y is the "reduced space" $\pi^{-1}(b)/\mathcal{F}$. **Key steps** to make sense of this ($\mathcal{F}$ need not be Lie, nor its action free or proper...):

1. Show directly that (as in Lie case) $\pi^{-1}(b)$ is a submanifold with

$$T_x(\pi^{-1}(b)) = \mathfrak{f}(x)^\sigma, \tag{*}$$

hence coisotropic with characteristic leaves the $\mathcal{F}$-orbits.

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Theorem (Z. [arXiv:1410.7950])

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⇓: Explicitly Y is the "reduced space" $\pi^{-1}(b)/\mathcal{F}$. **Key steps** to make sense of this ($\mathcal{F}$ need not be Lie, nor its action free or proper. . .):

2. Observe that the action $\mathcal{F} \longrightarrow \mathrm{Diff}(\pi^{-1}(b))$ factors into

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\text{onto by } (*)} & T^*_bB \\ f & \longmapsto & -D\dot{f}(b) \end{array} \quad \text{and} \quad \begin{array}{ccc} T^*_bB & \longrightarrow & \mathrm{Diff}(\pi^{-1}(b)) \\ a & \longmapsto & e^{\hat{a}} \end{array}$$

with $\hat{a}$ defined by $\sigma(\hat{a}(x), \cdot) = a \circ D\pi(x)$ $[T^*_bB \to T^*_xX \to T_xX]$. So its orbits are those of an action of the additive *Lie group* T^*_bB.

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$\Downarrow$: Explicitly Y is the "reduced space" $\pi^{-1}(b)/\mathcal{F}$. **Key steps** to make sense of this ($\mathcal{F}$ need not be Lie, nor its action free or proper. . .):

3. Observe that the moment map Φ relates $\hat{a} \in \mathrm{Vect}(\pi^{-1}(b))$ with the *constant* field $\check{a} \in \mathrm{ann}(\mathfrak{h}) \subset \mathfrak{g}^*$ defined by $\langle \check{a}, Z\rangle = \langle a, Z(b)\rangle$:

$$\langle D\Phi(x)(\hat{a}(x)), Z\rangle = \sigma(\hat{a}(x), Z(x)) = \langle a, D\pi(x)(Z(x))\rangle = \langle \check{a}, Z\rangle.$$

So Φ intertwines the $T^*_b B$ action with a mere translation action: $\Phi(e^{\hat{a}}(x)) = \Phi(x) + \check{a}$. Latter is free and proper $\Rightarrow$ so is former.

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Sample applications

Corollary 1 (Normal **abelian** subgroup analysis)

Suppose $\mathrm{X} = \mathrm{G}(x)$ *is a coadjoint orbit of* G *which has a closed connected normal* ***abelian*** *subgroup* A. *Then* $\mathrm{X} = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\,\mathrm{Y}$ *where* H *is the stabilizer in* G *of* $x_{|\mathfrak{a}} \in \mathfrak{a}^*$ *and* $\mathrm{Y} = \mathrm{H}(x_{|\mathfrak{h}})$.

Proof: $\{\langle\cdot, \mathrm{Z}\rangle : \mathrm{Z} \in \mathfrak{a}\}$ is a transitive system of imprimitivity for X. □

Corollary 2 (Symplectic Kirillov-Bernat)

Suppose $\mathrm{X} = \mathrm{G}(x)$ *is a coadjoint orbit of an* ***exponential*** *Lie group* G. *Then* X *is monomial, i.e.* G *admits a closed connected subgroup* P *such that* $\mathrm{X} = \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}\{x_{|\mathfrak{p}}\}$.

Note: This gives a new construction of *Pukánszky polarizations* $(\mathfrak{P})$.

Proof: Apply Corollary 1 inductively to normal abelian subgroups furnished by a lemma of Takenouchi [1957] to obtain

$$\mathrm{X} = \mathrm{Ind}_{\mathrm{H}_1}^{\mathrm{G}} \cdots \mathrm{Ind}_{\mathrm{H}_i}^{\mathrm{H}_{i-1}} \mathrm{Y}_i = \mathrm{Ind}_{\mathrm{H}_i}^{\mathrm{G}} \mathrm{Y}_i$$

(induction in stages) where the $\mathrm{Y}_i = \mathrm{H}_i(x_{|\mathfrak{h}_i})$ (H_i still exponential) have decreasing dimension, until Y_n is a point-orbit of $\mathrm{P} = \mathrm{H}_n$ say. □

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Contact imprimitivity

Contact imprimitivity theorem

Trying to mimick Mackey analysis for ***nonabelian*** normal subgroups soon forces one to remember that unitary representations really correspond to ***prequantum*** G-spaces $(\tilde{X}, \alpha)$.

These are principal circle bundles $P : \tilde{X} \to X$ with G-action preserving a connection 1-form α. The G-action, 2-form $d\alpha$, and 'contact' moment map $\langle \tilde{\Phi}(\tilde{x}), Z \rangle = \alpha(Z(\tilde{x}))$ descend and make X a hamiltonian G-space.

Suppose $(\tilde{Y}, \beta)$ is a prequantum H-space over Y. Then $(T^*G \times \tilde{Y}, \theta + \beta)$ is a prequantum $G \times H$-space over $T^*G \times Y$. Reducing by H produces an ***induced*** prequantum G-space $(\mathrm{Ind}_H^G \tilde{Y}, \alpha_{\mathrm{ind}})$ over $\mathrm{Ind}_H^G Y$ admitting a transitive ***system of imprimitivity***: a G-invariant, commutative Lie algebra $\tilde{\mathfrak{f}}_{\mathrm{ind}}$ of complete α-preserving vector fields. In fact $\tilde{\mathfrak{f}}_{\mathrm{ind}} \simeq \mathfrak{f}_{\mathrm{ind}}$.

Theorem

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End!