Primary Spaces, Mackey's Obstruction, and the Generalized Barycentric Decomposition[*]

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Gone Fishing 2012

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[*] arXiv:1203.5723, joint with Patrick Iglesias-Zemmour (Aix-Marseille).

Let G be a group and N a normal subgroup, so that

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1.$$

Questions about G often reduce to similar ones about N and G/N.

Classic example: Mackey Theory

To classify $\mathrm{Irr}(G) = \dfrac{\{X : \text{irreducible unitary G-module}\}}{\text{isomorphism}}$, 3 steps:

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3. Then $X = U \otimes T$, with N-action: (given) ⊗ (trivial)
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Now assume G and N are Lie groups. We want a parallel

Symplectic Mackey Theory

To classify $\dfrac{\{(X, \omega, \Phi) : \text{homogeneous Hamiltonian G-space}\}}{\text{isomorphism}}$, expect 3 steps:

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 & & \circlearrowright & \circlearrowright & \\
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sends N-orbit to N-orbit.

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A hamiltonian N-space $X = (X, \omega, \Pi)$ is ***primary*** if its moment map Π is onto a single coadjoint orbit U of N.
If N (or U) needs emphasis we say N-*primary* (*over* U).

Examples:

1. Any ***homogeneous*** hamiltonian N-space—since its moment map is an orbit covering, $\bar{U} \to U$.
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$K = N_c$

③ ***“Flat bundles”*:** Above U there a maximal homogeneous covering

$$\tilde{U} \xrightarrow{\rho} U,$$

namely $N/K^o \longrightarrow N/K$, with principal group

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So every time Γ acts on a symplectic manifold V while preserving its 2-form, we can form the associated bundle

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Theorem 1 (Generalized barycentric decomposition)

(1) Every primary N-space (X, ω, Π) over U is such a flat bundle. Indeed we always have

$$X = \tilde{U} \times_\Gamma V \qquad \text{where} \qquad V = \Pi^{-1}(c).$$

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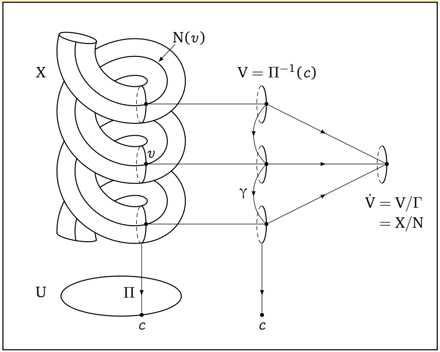

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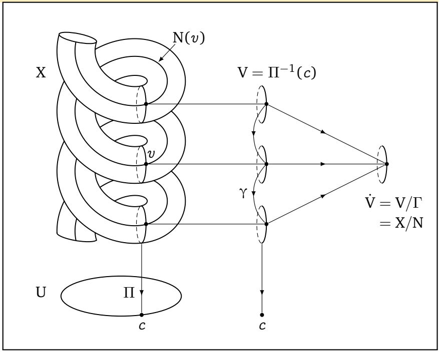

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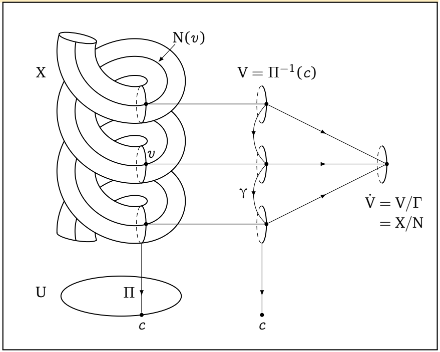

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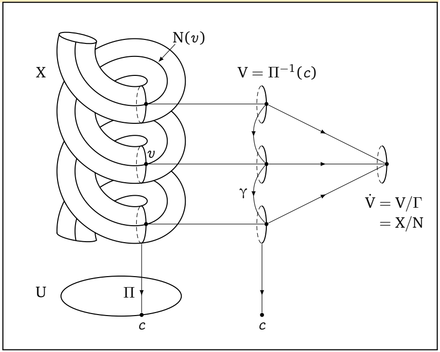

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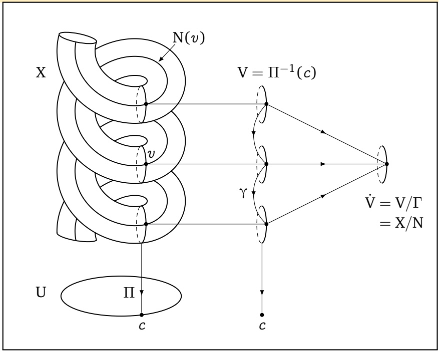

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A vast supply of examples: KKS reduced spaces

If (Y, σ, Ψ) is any hamiltonian N-space, then under appropriate transversality conditions one can form the *reduced space at* U,

$$X = \Psi^{-1}(U)/\ker(\sigma).$$

This is a primary N-space over U, hence a flat bundle as above.

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If $(\mathrm{Y}, \sigma, \Psi)$ is any hamiltonian N-space, then under appropriate transversality conditions one can form the *reduced space at* U,

$$\mathrm{X} = \Psi^{-1}(\mathrm{U})/\ker(\sigma).$$

This is a primary N-space over U, hence a flat bundle as above.

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$\quad\rotatebox{90}{$\in$}$
$U = N(c)$
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About Γ (N connected):

- N compact or exponential $\Rightarrow \Gamma = 0 \quad \Rightarrow X = U \times V$.
- N solvable $\Rightarrow \Gamma = \mathbf{Z}^d$.
- N semisimple $\Rightarrow \Gamma =$ finite product of $\mathbf{Z}_d, S_3, S_4, S_5$.
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- $\mathrm{U} = \mathbf{R} \times \mathrm{S}^1$, coadjoint orbit of $\mathrm{N} = \tilde{\mathrm{E}}(2)$. So $\tilde{\mathrm{U}} = \mathbf{R}^2$, $\Gamma = \mathbf{Z}$.
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Then $\mathrm{X} = \tilde{\mathrm{U}} \times_\Gamma \mathrm{V}$ doesn't split.

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So $\pi_1(\mathrm{X})$ is the Heisenberg group over $\mathbf{Z}$. That is impossible for the product of a cylinder (or plane) by any surface. □

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Other proof.

- If X did split *nontrivially* as $\bar{U} \times T$ then the fiber V would not be connected; but $\mathbf{T}^2$ is.
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Remark: Here V/Γ is not a manifold nor even an orbifold.

$(\mathfrak{n}^*/\mathrm{N})^{\mathrm{G}}$

$\rotatebox{90}{$\in$}$

$\mathrm{U} = \mathrm{N}(c)$

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N-primary G-spaces

Back with N normal in an ambient group G, we ask about N-primary spaces that arise by *restriction* of an action of the larger group:

$$\begin{tikzcd}
\mathrm{G} \arrow[dr, dashed, "\text{G-action}"] & \\
 & \mathrm{Diff}(\mathrm{X}, \omega) \\
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\end{tikzcd}
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Then we must have $\mathrm{U} \in (\mathfrak{n}^*/\mathrm{N})^{\mathrm{G}}$. So G acts on U, and the stabilizer $\mathrm{L} = \mathrm{G}_c$ acts on the fiber $\mathrm{V} = \Pi^{-1}(c)$. Our theorem becomes:

Theorem 2

(1) The bijection $\mathrm{X} \rightleftarrows \mathrm{V}$ of Theorem 1 induces another between

  (a) N-primary hamiltonian G-spaces $(\mathrm{X}, \omega, \Phi)$ over U;

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$$\begin{tikzcd} G \arrow[dr, dashed, "\text{G-action}"] & \\ & \mathrm{Diff}(X,\omega) \\ N \arrow[uu] \arrow[ur, "\text{N-action}"'] & \end{tikzcd} \qquad \begin{tikzcd} & \mathfrak{g}^* \arrow[d] \arrow[loop right, "G"] \\ X \arrow[ur, dashed, "\Phi"] \arrow[r, "\Pi"'] & \mathfrak{n}^* \arrow[loop right, "G"] \end{tikzcd}$$

Then we must have $U \in (\mathfrak{n}^*/N)^G$. So G acts on U, and the stabilizer $L = G_c$ acts on the fiber $V = \Pi^{-1}(c)$. Our theorem becomes:

$(\mathfrak{n}^*/N)^G$
$\rotatebox{90}{$\in$}$
$U = N(c)$
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Theorem 2

(1) The bijection $X \rightleftarrows V$ of Theorem 1 induces another between
  (*a*) N-primary hamiltonian G-spaces (X, ω, Φ) over U;
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(2) Two objects X_1, X_2 in (a) are isomorphic iff the corresponding $V_i = \Pi_i^{-1}(c)$ in (b) are isomorphic as hamiltonian L-spaces.

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$(a \to b)$ is easy (restrict the action and moment map of G to L).

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. . . $\tilde{\mathrm{G}}$ defined by

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \Delta(\mathrm{K}^{\mathrm{o}}) \arrow[r] \arrow[d] & \Delta(\mathrm{K}^{\mathrm{o}}) \arrow[r] \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \Delta(\mathrm{K}) \arrow[r] \arrow[d] & \mathrm{N} \rtimes \mathrm{L} \arrow[r, "\pi"] \arrow[d, "{[\![\cdot,\cdot]\!]}"] & \mathrm{G} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r, dashed] & \Gamma \arrow[r, dashed] \arrow[d] & \tilde{\mathrm{G}} \arrow[r, dashed] \arrow[d] & \mathrm{G} \arrow[r, dashed] \arrow[d] & 1 \\
 & 1 & 1 & 1, &
\end{tikzcd}$$

where $\Delta(k) = (k^{-1}, k)$ and $\pi(n, l) = nl$.

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Now $\tilde{\mathrm{G}}$ lifts to act on $\tilde{\mathrm{U}}$ and V by

$$[\![n, l]\!](\tilde{u}) = nl\tilde{u}l^{-1}, \qquad \text{resp.} \qquad [\![n, l]\!](v) = l(v)$$

with moment maps

$$\phi(\tilde{u}) = \tilde{u}(\check{c}), \qquad \text{resp.} \qquad \psi(v) = j(\Psi(v) - \check{c}_{|\mathfrak{l}})$$

where we have fixed an element $\check{c} \in \mathfrak{g}^*$ projecting to $c \in \mathfrak{n}^*$ and j denotes the isomorphism $\mathrm{ann}_{\mathfrak{l}^*}(\mathfrak{k}) \to \mathrm{ann}_{\mathfrak{g}^*}(\mathfrak{n})$ which exists because $\mathrm{G} = \mathrm{NL}$ implies $\mathrm{G}/\mathrm{N} = \mathrm{L}/\mathrm{K}$.

ϕ and ψ depend on the choice of $\check{c}$, but $\phi + \psi : \tilde{\mathrm{U}} \times \mathrm{V} \to \mathfrak{g}^*$ doesn't.

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Moreover one finds, miraculously,

$$\phi(\tilde{g}(\tilde{u})) = g(\phi(\tilde{u})) + \theta(\tilde{g})$$
$$\psi(\tilde{g}(v)) = g(\psi(v)) - \theta(\tilde{g})$$

where $\theta(\llbracket n, l \rrbracket) = \check{c} - l(\check{c})$. So the two moment maps are not in general equivariant but their sum is. Finally one checks (using $\Gamma \lhd \tilde{\mathrm{G}}$) that the diagonal action $\tilde{g}(\tilde{u}, v) = (\tilde{g}(\tilde{u}), \tilde{g}(v))$ on $\tilde{\mathrm{U}} \times \mathrm{V}$ and its moment map $\phi + \psi$ descend to the sought G-action and moment map on $\tilde{\mathrm{U}} \times_{\Gamma} \mathrm{V}$. ☐

Remark: N and L inject as subgroups $\tilde{\mathrm{N}} = \llbracket \mathrm{N}, e \rrbracket$ and $\tilde{\mathrm{L}} = \llbracket e, \mathrm{L} \rrbracket$ of $\tilde{\mathrm{G}} = \tilde{\mathrm{N}}\tilde{\mathrm{L}}$.

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$$\begin{aligned}\phi(\tilde{g}(\tilde{u})) &= g(\phi(\tilde{u})) + \theta(\tilde{g})\\ \psi(\tilde{g}(v)) &= g(\psi(v)) - \theta(\tilde{g})\end{aligned}$$

where $\theta([\![n, l]\!]) = \check{c} - l(\check{c})$. So the two moment maps are not in general equivariant but their sum is. Finally one checks (using $\Gamma \triangleleft \tilde{G}$) that the diagonal action $\tilde{g}(\tilde{u}, v) = (\tilde{g}(\tilde{u}), \tilde{g}(v))$ on $\tilde{U} \times V$ and its moment map $\phi + \psi$ descend to the sought G-action and moment map on $\tilde{U} \times_\Gamma V$. $\square$

Remark: N and L inject as subgroups $\tilde{N} = [\![N, e]\!]$ and $\tilde{L} = [\![e, L]\!]$ of $\tilde{G} = \tilde{N}\tilde{L}$.

$(\mathfrak{n}^*/\mathrm{N})^{\mathrm{G}}$
$\rotatebox{90}{$\in$}$
$\mathrm{U} = \mathrm{N}(c)$
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(1) The bijection $\mathrm{X} \rightleftarrows \mathrm{V}$ of Theorem 1 induces another between

- (*a*) N-primary hamiltonian G-spaces $(\mathrm{X}, \omega, \Phi)$ over U;
- (*b*) K-primary hamiltonian L-spaces $(\mathrm{V}, \omega_{\mathrm{V}}, \Psi)$ over $\{c_{|\mathfrak{k}}\}$.

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$(\mathfrak{n}^*/N)^G$

$\rotatebox{90}{$\in$}$

$U = N(c)$

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$L = G_c$

Corollary 1 (Generalized Kœnig Theorem)

Not only does G act on $X = \tilde{U} \times_\Gamma V$, but the larger group $\tilde{G} \times \tilde{G}$ acts factor-wise on $\tilde{U} \times V$ with moment map (ϕ, ψ) such that

$$\Phi([\tilde{u}, v]) = \phi(\tilde{u}) + \psi(v).$$

The second action is really an action of $\tilde{G}/\tilde{N}$ with moment map $\psi : V \to \mathrm{ann}_{\mathfrak{g}^*}(\mathfrak{n})$.

Corollary 2 (of proof)

Attached to each $U \in (\mathfrak{n}^*/N)^G$ is a well-defined cohomology class $[\theta] \in H^1(\tilde{G}/\tilde{N}, (\mathfrak{g}/\mathfrak{n})^*)$ which measures the obstruction to making U a hamiltonian G-space, and vanishes if $c_{|\mathfrak{k}} = 0$. If $c_{|\mathfrak{k}} \neq 0$, then $[\langle D\theta(e)(\cdot), \cdot\rangle] \in H^2(\mathfrak{g}/\mathfrak{n}, R)$ is the class of the central extension

$$0 \longrightarrow \mathfrak{k}/\mathfrak{j} \longrightarrow \mathfrak{l}/\mathfrak{j} \longrightarrow \mathfrak{l}/\mathfrak{k} \longrightarrow 0 \qquad (*)$$

where $\mathfrak{j} = \ker(c_{|\mathfrak{k}})$.

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$\mathrm{U} = \mathrm{N}(c)$

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Definition

We call this extension (*) the ***infinitesimal Mackey obstruction*** of U (relative to G).

Remark: When U is *integral*, i.e. K admits a character χ with differential $ic_{|\mathfrak{k}}$, (*) integrates to a group extension

$$1 \longrightarrow \mathrm{K}/\mathrm{J} \longrightarrow \mathrm{L}/\mathrm{J} \longrightarrow \mathrm{L}/\mathrm{K} \longrightarrow 1$$

where $\mathrm{J} = \ker(\chi)$. This is precisely the Mackey obstruction found by Auslander-Kostant and Duflo for the representation “quantizing” (U, χ) (N nilpotent, G solvable).

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A non-split example

$$\mathrm{N} = \left\{ n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & \mathrm{e}^{2\pi i a} & 0 & b \\ & & & 1 & a \\ & & & & 1 \end{pmatrix} : \begin{array}{l} a \in \mathbf{R} \\ b \in \mathbf{C} \end{array} \right\}$$

$$\mathrm{N} = \{g : c = e = 0\}.$$

Identify $\mathfrak{g}^*$ with $\mathrm{R} \times \mathbf{C} \times \mathrm{R}^3$ by $(p, z, r, s, t) =$ value of the 1-form

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A non-split example

$$\mathrm{G}=\left\{ g=\begin{pmatrix}1 & c & 0 & e & f\\ & 1 & 0 & 0 & e\\ & & \mathrm{e}^{2\pi i a} & 0 & b\\ & & & 1 & a\\ & & & & 1\end{pmatrix} : \begin{array}{l} a,c,e,f\in\mathbf{R}\\ b\in\mathbf{C}\end{array}\right\}$$

$$\mathrm{N}=\{g : c=e=0\}.$$

Identify $\mathfrak{g}^*$ with $\mathbf{R}\times\mathbf{C}\times\mathbf{R}^3$ by (p,z,r,s,t) = value of the 1-form

$$p\,da+\mathrm{Re}(\bar{z}\,db)-r\,dc-s\,de-t\,df$$

at the identity. So $\mathfrak{g}^*\to\mathfrak{n}^*$ writes $(p,z,r,s,t)\mapsto(p,z,t)$. The orbit $\mathrm{X}=\mathrm{G}(0,1,0,0,1)$ is N-primary over U with fiber V, where

$$\begin{aligned}\mathrm{X}&=\big\{\big(p,\mathrm{e}^{2\pi i q},r,s,1\big) : p,q,r,s\in\mathbf{R}\big\},\\ \mathrm{U}&=\big\{\big(p,\mathrm{e}^{2\pi i q},1\big) : p,q\in\mathbf{R}\big\}, & \omega_{\mathrm{U}}&=dp\wedge dq,\\ \mathrm{V}&=\big\{\big(0,1,r,s,1\big) : r,s\in\mathbf{R}\big\}, & \omega_{\mathrm{V}}&=dr\wedge ds.\end{aligned}$$

We claim that X does *not* split as $U \times V$ (or otherwise). A first hint of this is that $\omega_X = dp \wedge dq + dq \wedge dr + dr \wedge ds \neq \omega_U + \omega_V$.

$(\mathfrak{n}^*/N)^G$

$\rotatebox[origin=c]{-90}{$\in$}$

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Proof. Since the fiber V is connected, it is enough to see that Γ acts nontrivially on it. But one finds

$$g\begin{pmatrix} p \\ z \\ r \\ s \\ 1 \end{pmatrix} = \begin{pmatrix} p + e + \mathrm{Re}\left(\overline{2\pi i b}\, e^{2\pi i a} z\right) \\ e^{2\pi i a} z \\ r + e \\ s + a - c \\ 1 \end{pmatrix}$$

and $n\left(\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right) = \begin{pmatrix} \mathrm{Re}(\overline{2\pi i b}\, e^{2\pi i a}) \\ e^{2\pi i a} \\ 1 \end{pmatrix}$. So $K = N_{\left(\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right)} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & f \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & b \\ & & & 1 & a \\ & & & & 1 \end{pmatrix} : \begin{smallmatrix} a \in \mathbb{Z} \\ b, f \in \mathbb{R} \end{smallmatrix} \right\}$

and $\Gamma = K/K^o = \mathbb{Z}$ acts nontrivially on V. ☐

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We claim that X does *not* split as U × V (or otherwise). A first hint of this is that $\omega_{\mathrm{X}} = dp \wedge dq + dq \wedge dr + dr \wedge ds \neq \omega_{\mathrm{U}} + \omega_{\mathrm{V}}$.

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Proof. Since the fiber V is connected, it is enough to see that Γ acts nontrivially on it. But one finds

$$k\begin{pmatrix}0\\1\\r\\s\\1\end{pmatrix} = \begin{pmatrix}0\\1\\r\\s+a\\1\end{pmatrix}$$

and $n\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}\mathrm{Re}(\overline{2\pi i b}\,\mathrm{e}^{2\pi i a})\\ \mathrm{e}^{2\pi i a}\\ 1\end{pmatrix}$. So $\mathrm{K} = \mathrm{N}_{\begin{pmatrix}0\\1\\1\end{pmatrix}} = \left\{\begin{pmatrix}1&0&0&0&f\\ &1&0&0&0\\ &&1&0&b\\ &&&1&a\\ &&&&1\end{pmatrix} : \begin{array}{c}a\in\mathbf{Z}\\ b,f\in\mathbf{R}\end{array}\right\}$

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