Quantization, after Souriau

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Geometric Quantization: Old and New
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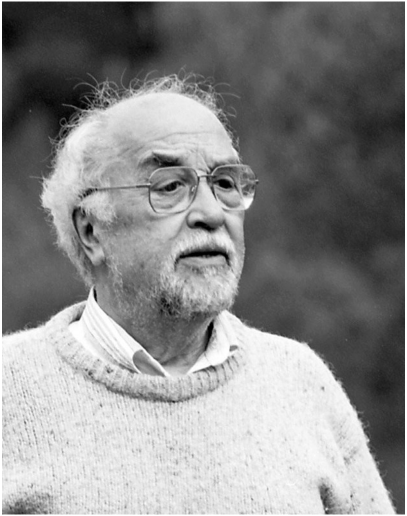
Abstract: J.-M. Souriau spent the years 1960-2000 in a uniquely dogged inquiry into what exactly quantization is and isn't. I will report on results (of arXiv:1310.7882 etc.) pertaining to the last (still unsatisfactory!) formulation he gave.

J.-M. Souriau



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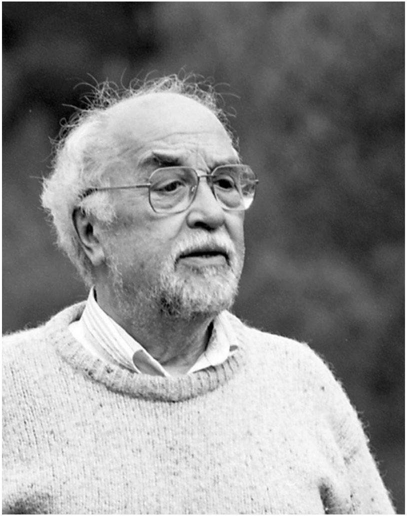

What is quantization?

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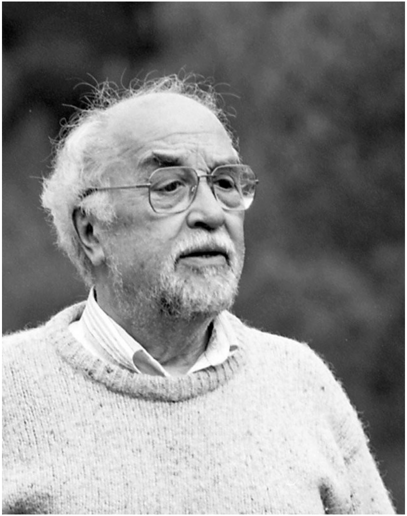

What is quantization?

« How do I arrive at the matrix that represents a given quantity in a system of known constitution? »

—H. Weyl, *Quantenmechanik und Gruppentheorie* (1927)

Prequantization

Let (X, ω) be a prequantizable symplectic manifold: $[\omega] \in H^2(X, \mathbf{Z})$.

Mantra:

Prequantization constructs a representation of the Poisson algebra $C^\infty(X)$, which is "too large" because not irreducible enough.

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Souriau:

Not the point! What prequantization constructs is a group $\mathrm{Aut}(L)$ with “Lie algebra” $C^\infty(X)$, *of which* X *is a coadjoint orbit*.

(Every prequantizable symplectic manifold is a coadjoint orbit, 1985.)

Quantization?

Mantra:

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Besides, it doesn't exist ("by van Hove's no-go theorem").

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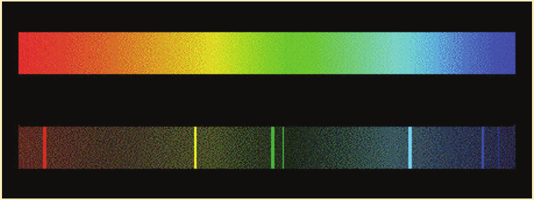
Souriau:

No! Quantization is a switch from *classical states* to *quantum states*:



Group algebra. States

- $\mathbf{C}[G] :=$ {finitely supported functions $G \to \mathbf{C}$} $\ni c = \sum_{g \in G} c_g \delta^g$ is a $*$-algebra: $\delta^g \cdot \delta^h = \delta^{gh}$, $(\delta^g)^* = \delta^{g^{-1}}$ (and a G-module)
- $\mathbf{C}[G]' \cong \mathbf{C}^G =$ {all functions $m : G \to \mathbf{C}$}: $\langle m, c \rangle = \sum c_g m(g)$
- G-invariant sesquilinear forms on $\mathbf{C}[G]$ write $(c, d) \mapsto \langle m, c^* \cdot d \rangle$

$$(\delta^e, g\delta^e) \mapsto m(g)$$

Definition, Theorem (GNS, L. Schwartz)

Call m a *state of* G if positive definite: $\langle m, c^* \cdot c \rangle \geqslant 0$, and $m(e) = 1$.

- Then $\overline{\mathbf{C}[G]/\mathbf{C}[G]^\perp}$ is a unitary G-module, realizable in $\mathbf{C}[G]'$ as

$$\mathrm{GNS}_m = \left\{ \varphi \in \mathbf{C}^G \text{ such that } \|\varphi\|^2 := \sup_{c \in \mathbf{C}[G]} \frac{|\langle \overline{\varphi}, c \rangle|^2}{\langle m, c^* \cdot c \rangle} < \infty \right\}.$$

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Group algebra. States

Example 1: Characters

If $\chi : G \to U(1)$ is a character, then χ is a state and

$$\mathrm{GNS}_\chi = \mathbf{C}_\chi$$

($= \mathbf{C}$ where G acts by χ).

Example 2: Discrete induction (Blattner 1963)

If n is a state of a subgroup $H \subset G$ and $m(g) = \begin{cases} n(g) & \text{if } g \in H, \\ 0 & \text{otherwise,} \end{cases}$ then m is a state and

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If n is a state of a subgroup $\mathrm{H} \subset \mathrm{G}$ and $m(g) = \begin{cases} n(g) & \text{if } g \in \mathrm{H}, \\ 0 & \text{otherwise,} \end{cases}$ then m is a state and

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Classical (statistical) states

Let X be a coadjoint orbit of G (say a Lie group). Continuous states m of $(\mathfrak{g}, +)$ correspond to probability measures μ on $\mathfrak{g}^*$ (Bochner):

$$m(Z) = \int_{\mathfrak{g}^*} e^{i\langle x, Z\rangle} d\mu(x). \tag{1}$$

Definition

A ***statistical state*** for X is a state m of $\mathfrak{g}$ which is concentrated on X, in the sense that its spectral measure (μ above) is.

This works even without assuming continuity of m: in (1), make $\mathfrak{g}$ discrete and hence replace $\mathfrak{g}^*$ by its *Bohr compactification*

$$\hat{\mathfrak{g}} = \{\textit{all} \text{ characters of } \mathfrak{g}\},$$

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Let X be a coadjoint orbit of G (say a Lie group).

Definition (*equivalent to* Souriau's)

A ***quantum state*** for X is a state m of G, such that for every abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, the state $m \circ \exp_{|\mathfrak{a}}$ of $\mathfrak{a}$ is concentrated on $b\mathrm{X}_{|\mathfrak{a}}$.

Statistical interpretation: the spectral measure of $m \circ \exp_{|\mathfrak{a}}$ gives the probability distribution of $x_{|\mathfrak{a}}$ (or "joint probability" of the Poisson commuting functions $\langle \cdot, \mathrm{Z}_j \rangle$ for Z_j in a basis of $\mathfrak{a}$).

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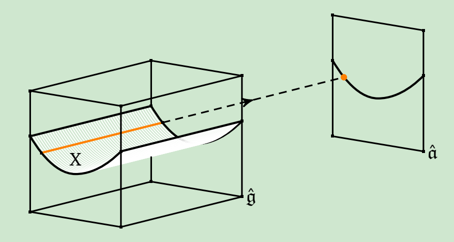

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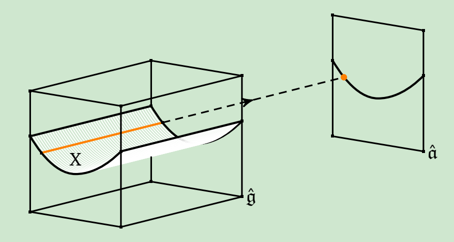

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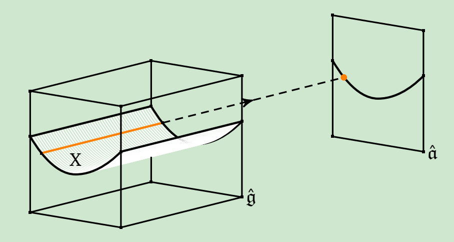

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If $V = GNS_m$, then $(\varphi, \cdot\, \varphi)$ is a quantum state for X for *all* unit $\varphi \in V$.

Definition

G-modules V with this property are ***quantum representations*** for X.

They need not be continuous, nor irreducible on transitive subgroups.

Example 1: Point-orbits

Suppose a state n of a connected Lie group H is quantum for a *point-orbit* $\{y\} \subset (\mathfrak{h}^*)^{H}$. Then y is *integral*, and n is the character such that

$$n(\exp(Z)) = e^{i\langle y, Z\rangle}. \tag{2}$$

A representation of H is quantum for $\{y\}$ iff it is a *multiple* of this n.

We will call states of $G \supset H$ that restrict to (2) ***eigenstates belonging to*** $y \in (\mathfrak{h}^*)^{H}$ — or by abuse, to the (generically *coisotropic*) preimage of y in some $X \subset \mathfrak{g}^*$. Weinstein (1982) called attaching waves to lagrangian submanifolds the FUNDAMENTAL QUANTIZATION PROBLEM.

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Let L be the prequantization line bundle over $X = (\mathbf{R}^2, dp \wedge dq)$.
The resulting representation of Aut(L) in $L^2(X)$ is not quantum for X.

Sketch of proof:
It represents the flow of the *bounded* hamiltonian $H(p,q) = \sin p$ by a 1-parameter group whose self-adjoint generator is *unbounded* — it's equivalent to multiplication by $\sin p + (k-p)\cos p$ in $L^2(\mathbf{R}^2, dp\,dk)$.

Remark

We are rejecting this representation for *spectral* reasons. Unlike van Hove who rejected it for being *reducible* on the Heisenberg subgroup, we can still hope that Aut(L) has a representation quantizing X.

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Theorem (Howe-Z., *Ergodic Theory Dynam. Systems* 2015)

- G *noncompact simple: every nonzero coadjoint orbit has* $b\mathrm{X} = b\mathfrak{g}^*$.
- G *connected nilpotent: every coadjoint orbit has the same Bohr closure as its affine hull.*

Corollary

- G *noncompact simple: every unitary representation is quantum for every nonzero coadjoint orbit.*
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(a) irreducible $\Leftrightarrow$ H is a *polarization at x* (: subordinate subgroup such that the bound $\dim(\mathrm{G}/\mathrm{H}) \geqslant \frac{1}{2}\dim(\mathrm{X})$ is attained);

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Remark: The coadjoint orbit $\mathrm{G}(x) = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{G}}\{x_{|\mathfrak{q}}\}$ (symplectic induction).

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$\mathfrak{g}^*$: identified with $\mathfrak{g}$ by means of the trace form $\langle \mathrm{Z}, \mathrm{Z}' \rangle = \mathrm{Tr}(\mathrm{Z}\mathrm{Z}')$
x : hyperbolic element of $\mathfrak{g}^*$ (: diagonalizable with real eigenvalues)
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χ : a character of the parabolic $\mathrm{Q} = \mathrm{G}_x \exp(\mathfrak{u})$ with differential $\mathrm{i}x_{|\mathfrak{q}}$.

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Theorem: The conjecture is true for $G = \mathbf{SL}_2(\mathbf{R})$ or $\mathbf{SL}_3(\mathbf{R})$, Q Borel.

Euclid's group $\mathrm{G} = \left\{ g = \begin{pmatrix} \mathrm{A} & c \\ 0 & 1 \end{pmatrix} : \begin{smallmatrix} \mathrm{A} \in \mathbf{SO}(3) \\ c \in \mathbf{R}^3 \end{smallmatrix} \right\}$

Example: TS^2

G acts naturally and symplectically on the manifold $\mathrm{X} \simeq \mathrm{TS}^2$ of oriented lines (a.k.a. light rays) in $\mathbf{R}^3$. 2-form$_{k,s}$:

$$\omega = k\, d\langle u, dr\rangle + s\, \mathrm{Area}_{S^2}.$$

The moment map

$$\Phi(u, r) = \begin{pmatrix} r \times ku + su \\ ku \end{pmatrix}$$

makes X into a coadjoint orbit of G.

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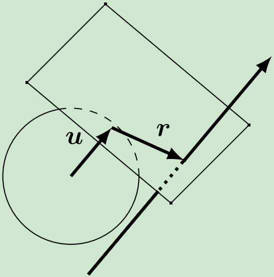

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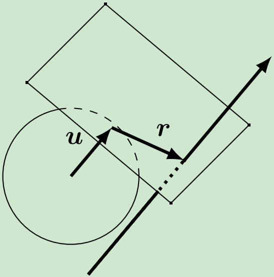

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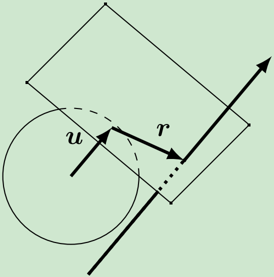

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Case $s = 0$:

We have unique* eigenstates belonging to 3 types of lagrangians:

	$m\begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{[illegible]} & [illegible] \\ 0 & \text{otherwise} \end{cases}$

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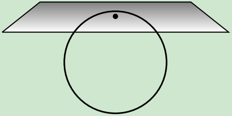 tangent space at $\boldsymbol{n} = \boldsymbol{e}_3$	$m\begin{pmatrix} \mathrm{A} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{i\langle \boldsymbol{n}, k\boldsymbol{c}\rangle} & \text{if } \mathrm{A}\boldsymbol{n} = \boldsymbol{n}, \\ 0 & \text{otherwise} \end{cases}$
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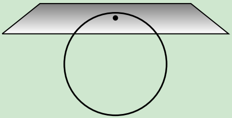 tangent space at $\boldsymbol{n} = \boldsymbol{e}_3$	$m\begin{pmatrix} A & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{i\langle \boldsymbol{n}, k\boldsymbol{c}\rangle} & \text{if } A\boldsymbol{n} = \boldsymbol{n}, \\ 0 & \text{otherwise} \end{cases}$
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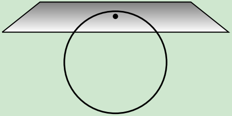
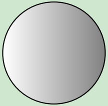
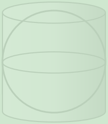
 tangent space at $\boldsymbol{n} = \boldsymbol{e}_3$	$m\begin{pmatrix} A & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{i\langle \boldsymbol{n}, k\boldsymbol{c}\rangle} & \text{if } A\boldsymbol{n} = \boldsymbol{n}, \\ 0 & \text{otherwise} \end{cases}$
 zero section	$m\begin{pmatrix} A & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \dfrac{\sin \lVert k\boldsymbol{c} \rVert}{\lVert k\boldsymbol{c} \rVert}$
 equator's normal bundle	$m\begin{pmatrix} A & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{cases} J_0(\lVert k\boldsymbol{c}_\perp \rVert) & \text{if } A\boldsymbol{n} = \pm\boldsymbol{n}, \\ 0 & \text{otherwise} \end{cases}$

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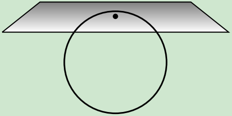
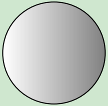
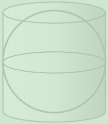
 tangent space at $\boldsymbol{n} = \boldsymbol{e}_3$	$m\begin{pmatrix} \mathrm{A} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{i\langle \boldsymbol{n}, k\boldsymbol{c}\rangle} & \text{if } \mathrm{A}\boldsymbol{n} = \boldsymbol{n}, \\ 0 & \text{otherwise} \end{cases}$
 zero section	$m\begin{pmatrix} \mathrm{A} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \dfrac{\sin \lVert k\boldsymbol{c} \rVert}{\lVert k\boldsymbol{c} \rVert}$
 equator's normal bundle	$m\begin{pmatrix} \mathrm{A} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{cases} \mathrm{J}_0(\lVert k\boldsymbol{c}_\perp \rVert) & \text{if } \mathrm{A}\boldsymbol{n} = \pm\boldsymbol{n}, \\ 0 & \text{otherwise} \end{cases}$

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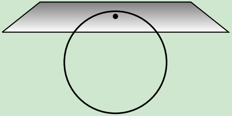
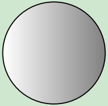
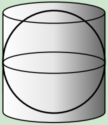
 tangent space at $\boldsymbol{n} = \boldsymbol{e}_3$	$m\begin{pmatrix} \mathrm{A} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{i\langle \boldsymbol{n}, k\boldsymbol{c}\rangle} & \text{if } \mathrm{A}\boldsymbol{n} = \boldsymbol{n}, \\ 0 & \text{otherwise} \end{cases}$
 zero section	$m\begin{pmatrix} \mathrm{A} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \dfrac{\sin \lVert k\boldsymbol{c} \rVert}{\lVert k\boldsymbol{c} \rVert}$
 equator's normal bundle	$m\begin{pmatrix} \mathrm{A} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{cases} \mathrm{J}_0(\lVert k\boldsymbol{c}_\perp \rVert) & \text{if } \mathrm{A}\boldsymbol{n} = \pm\boldsymbol{n}, \\ 0 & \text{otherwise} \end{cases}$

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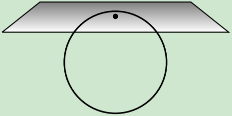
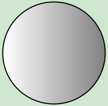
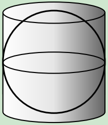
 tangent space at $\boldsymbol{n} = \boldsymbol{e}_3$	$m\begin{pmatrix} A & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{i\langle \boldsymbol{n}, k\boldsymbol{c}\rangle} & \text{if } A\boldsymbol{n} = \boldsymbol{n}, \\ 0 & \text{otherwise} \end{cases}$
 zero section	$m\begin{pmatrix} A & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \dfrac{\sin \lVert k\boldsymbol{c} \rVert}{\lVert k\boldsymbol{c} \rVert}$
 equator's normal bundle	$m\begin{pmatrix} A & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{cases} J_0(\lVert k\boldsymbol{c}_\perp \rVert) & \text{if } A\boldsymbol{n} = \pm\boldsymbol{n}, \\ 0 & \text{otherwise} \end{cases}$

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The resulting GNS modules can be realized as solution spaces of the Helmholtz equation

$$\Delta\psi + k^2\psi = 0 \tag{3}$$

with *scalar field* G-action $(g\psi)(\boldsymbol{r}) = \psi(A^{-1}(\boldsymbol{r} - \boldsymbol{c}))$ and cyclic vectors:

"plane wave"	"spherical wave"	"cylindrical wave"
$\psi(\boldsymbol{r}) = e^{-ikz}$	$\psi(\boldsymbol{r}) = \dfrac{\sin\lVert k\boldsymbol{r}\rVert}{\lVert k\boldsymbol{r}\rVert}$	$\psi(\boldsymbol{r}) = J_0(\lVert k\boldsymbol{r}_\perp\rVert)$

Challenge: Find all unitarizable G-modules of solutions of (3).

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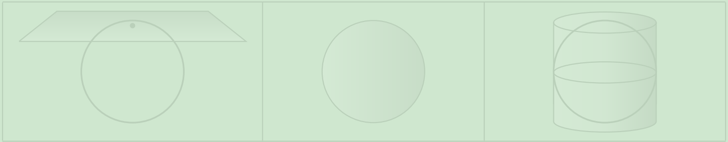

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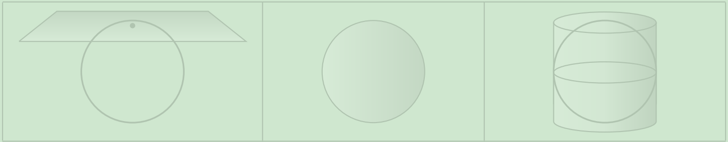

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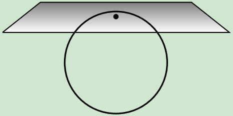	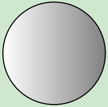	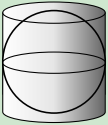
"plane wave" $\psi(\boldsymbol{r}) = \mathrm{e}^{-ikz}$	"spherical wave" $\psi(\boldsymbol{r}) = \dfrac{\sin\lVert k\boldsymbol{r}\rVert}{\lVert k\boldsymbol{r}\rVert}$	"cylindrical wave" $\psi(\boldsymbol{r}) = \mathrm{J}_0(\lVert k\boldsymbol{r}_\perp\rVert)$

Challenge: Find all unitarizable G-modules of solutions of (3).

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Case $s = 0$:

The resulting GNS modules can be realized as solution spaces of the Helmholtz equation

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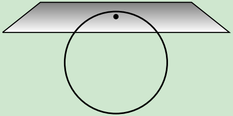	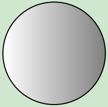	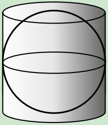
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Case $s = 1$ (zero section is no longer lagrangian):

The unique eigenstate belonging to the tangent space at $\boldsymbol{n}$ becomes

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one obtains a Hilbert space of almost-periodic solutions $\mathbf{F} = \mathbf{B} + i\mathbf{E}$ of the reduced Maxwell equations

$$\operatorname{div} \mathbf{F} = 0, \qquad \mathbf{curl}\, \mathbf{F} = k\mathbf{F},$$

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