Symplectic Mackey Theory*

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Abstract: Guillemin & Sternberg used symplectic induction to give a "Mackey-Wigner" description of Hamiltonian G-spaces when G has a ***normal abelian semidirect factor*** N. I will describe how this generalizes to a full "Mackey" description (and classification) valid for ***arbitrary normal*** subgroups N, and explain why this is best done in the setting of *prequantum* (contact) G-spaces.

*http://arxiv.org/abs/1410.7950

Let G be a group and N a normal subgroup, so that

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1.$$

Questions about G often reduce to similar ones about N and G/N.

Classic example: Mackey Theory

To classify $\mathrm{Irr}(G) = \dfrac{\{X : \text{irreducible unitary } G\text{-module}\}}{\text{isomorphism}}$, 3 steps:

In short, $X = \mathrm{Ind}_{G_U}^{G}(U \otimes V)$ and finding Irr(G) reduces to finding Irr(N) and *projective* representations of *subgroups of* G/N.

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2. Then $X = \mathrm{Ind}_{G_U}^{G} Y$, where $Y \in \mathrm{Irr}(G_U)$ is N-*primary* (: as N-module it involves U alone).
3. Then $Y = U \otimes V$, with N-action: (given)⊗(trivial)
   G_U-action: (*projective*)⊗(*projective*).

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- Motivation
- Symplectic induction
- Mackey-Wigner
- Kirillov-Bernat
- Imprimitivity
- Mackey theory
- Contact imprimitivity

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3. Then $Y = U \otimes V$, with N-actions

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Symplectic induction (G: Lie group)

Given a closed subgroup $H \subset G$ and a hamiltonian H-space (Y, τ, Ψ), Kazhdan-Kostant-Sternberg (1978) produce a hamiltonian G-space

$$(\mathrm{Ind}_H^G Y, \sigma_{\mathrm{ind}}, \Phi_{\mathrm{ind}})$$

as follows:

1. Endow $M := T^*G \times Y$ with the 2-form $\omega = d\theta + \tau$, $\theta =$ "$\langle p, dq \rangle$".
2. Let H act on M by $h(p, y) = (ph^{-1}, h(y))$.
3. This has moment map $\psi(p, y) = \Psi(y) - q^{-1}p_{|\mathfrak{h}}$ $(p \in T^*_qG)$.
4. Define $\mathrm{Ind}_H^G Y := \psi^{-1}(0)/H$ (Marsden-Weinstein subquotient).
5. The G-action $g(p, y) = (gp, y)$ and moment map $\phi(p, y) = pq^{-1}$ pass to the quotient, whence the claimed G-space structure.

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Proposition (Elementary properties)

1. $\dim(\mathrm{Ind}_H^G Y) = 2\dim(G/H) + \dim(Y)$.
2. M *meets* $\mathrm{Im}(\Phi_{\mathrm{ind}}) \Leftrightarrow M_{|\mathfrak{h}}$ *meets* $\mathrm{Im}(\Psi)$ $\quad (M \in \mathfrak{g}^*/G)$ (*Frobenius*).
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Theorem (Guillemin-Sternberg 1983++)

Let $\mathrm{N} \subset \mathrm{G}$ *be a closed connected normal* ***abelian*** *subgroup. Pick* $u \in \mathfrak{n}^*$ *and write* $\mathrm{H} = \mathrm{G}_u$. *Then* $\mathrm{Y} \mapsto \mathrm{X} = \mathrm{Ind}_\mathrm{H}^\mathrm{G} \mathrm{Y}$ *defines a bijection between*

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Crux of proof. Suppose $\mathrm{Y} = \mathrm{H}(x_{|\mathfrak{h}})$. Clearly $\mathrm{Y}_{|\mathfrak{n}} = \{u\}$. Want: Φ_{ind} 1-1 onto X. Now, we have

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Indeed: $\langle \mathfrak{n}(x), \mathrm{Z}\rangle = \langle x, [\mathfrak{n}, \mathrm{Z}]\rangle = \langle u, [\mathfrak{n}, \mathrm{Z}]\rangle = \langle \mathrm{Z}(u), \mathfrak{n}\rangle$ shows $\mathrm{ann}(\mathfrak{n}(x)) = \mathfrak{g}_u = \mathfrak{h}$.

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Indeed, for $\mathrm{Z} \in \mathfrak{n}$: $\langle \exp(\mathrm{Z})(x), \mathrm{Z}' \rangle = \left\langle x, \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathrm{ad}(\mathrm{Z})^n(\mathrm{Z}') \right\rangle$ $= \langle x, \mathrm{Z}' - [\mathrm{Z}, \mathrm{Z}'] \rangle = \langle x + \mathrm{Z}(x), \mathrm{Z}' \rangle$. So $\mathrm{N}(x) \supset x + \mathfrak{n}(x)$.

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So $\mathrm{M}_{|\mathfrak{h}}$ meets $\mathrm{Y} \Rightarrow \mathrm{M}$ meets X, and $\mathrm{Im}(\Phi_{\mathrm{ind}}) = \mathrm{X}$ by Frobenius.

Symplectic Mackey-Wigner

Example: Poincaré orbits

Consider

$$G = \left\{ g = \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} : \begin{array}{l} L \in \mathbf{SO}(3,1)^o \\ C \in \mathbf{R}^{3,1} \end{array} \right\},$$

semidirect product of $N = \mathbf{R}^{3,1}$ ($L = \mathbf{1}$) with the Lorentz group ($C = 0$). Then $\mathfrak{n}^*$ identifies with $\mathbf{R}^{3,1}$ where G acts by $g(P) = LP$. The theorem classifies the coadjoint orbits X of G in terms of the possible orbits $X_{|\mathfrak{n}} \ni u$ and Y, thus:

This is of course completely parallel with the representation theory of G as worked out by Wigner (1939).

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Application: Symplectic Kirillov-Bernat theory

Theorem (Z.)

Let G *be an exponential Lie group* (: $\exp$ *is a diffeomorphism* $\mathfrak{g} \to \mathrm{G}$) *and let* $\mathrm{X} = \mathrm{G}(x)$ *be a coadjoint orbit of* G. *Then* X *is monomial, i.e.* G *admits a closed connected subgroup* H, *such that*

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Symplectic Mackey Theory

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Imprimitivity (motivation)

Let G be a locally compact group (e.g. Lie), X a unitary G-module.

Definition

- A *system of imprimitivity* for X is a G-invariant, commutative C^*-subalgebra $A \subset \mathrm{End}(X)$.
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Remark: The *Gelfand transform* $a \mapsto \hat{a}$, defined by $\hat{a}(b) = b(a)$, is an isomorphism $A \to C_0(B)$. Its inverse E is a *-representation of $C_0(B)$ in X such that

$$E(f \circ g_B^{-1}) = g_X E(f) g_X^{-1},$$

i.e. a “system of imprimitivity” in the original Mackey-Blattner sense.

Imprimitivity (motivation)

The point of this is:

Theorem (Frobenius, Mackey)

The following are equivalent:

- X *admits a transitive system of imprimitivity with base* $B = G/H$ $(H = G_b$ *say*$)$;
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Let (X, σ, Φ) be a hamiltonian G-space.

Definition

- A *system of imprimitivity* for X is a G-invariant, commutative Lie subalgebra $\mathfrak{f} \subset C^\infty(X)$, such that the hamiltonian vector field $\operatorname{drag} f$ is complete for all $f \in \mathfrak{f}$.
- Its *base* is the image B of the "moment map" $\alpha : X \to \mathfrak{f}^*$, $\langle \alpha(x), f \rangle = f(x)$. Each $f \in \mathfrak{f}$ descends to a function $\bar{f}$ on B.

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Symplectic imprimitivity

Let (X, σ, Φ) be a hamiltonian G-space.

Definition

- A ***system of imprimitivity*** for X is a G-invariant, commutative Lie subalgebra $\mathfrak{f} \subset C^\infty(X)$, such that the hamiltonian vector field drag f is complete for all $f \in \mathfrak{f}$.
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Explanation: $\mathcal{F} :=$ ($\mathfrak{f}$ as an additive group) acts on X by $f_X = e^{\operatorname{drag} f}$ and π is *formally* a moment map for this action: $\operatorname{drag}\langle \pi(\cdot), f \rangle = \operatorname{drag} f$. Stabilizers G_b are *closed* so B's homogeneous structure is well-defined.

Symplectic imprimitivity theorem

Theorem (Z.)

The following are equivalent for a hamiltonian G*-space* (X, σ, Φ)*:*

- X *admits a transitive system of imprimitivity with base* $B = G/H$ $(H = G_b$ *say*$)$;
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$\Uparrow$: Recall, $\mathrm{Ind}_H^G Y = (T^*G \times Y)//H$. Now a G-equivariant projection

$$\pi_{\mathrm{ind}} : \mathrm{Ind}_H^G Y \to G/H$$

arises by noting that the map $T^*G \times Y \to G/H$ sending $T_q^*G \times Y$ to qH is constant on H-orbits, hence passes to the (sub)quotient. Then one checks that

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Explanation:

$\Downarrow$: *Formally* this is Mackey-Wigner applied to the group $\mathcal{F} \rtimes G$ and abelian normal subgroup $\mathcal{F}$. Explicitly Y is the "reduced space" $\pi^{-1}(b)/\mathcal{F}$. Proof subtler as $\mathcal{F}$ need not be Lie, nor its action free or proper...

Little group step

From now on, suppose $N \subset G$ is a closed normal subgroup. Then

- G acts naturally on N and $\mathfrak{n}$ by conjugation.
- G acts naturally on $\mathfrak{n}^*$ by contragredience.
- G respects the partition of $\mathfrak{n}^*$ into (coadjoint) N-orbits.
- So G acts in the orbit space $\mathfrak{n}^*/N$, and we have a composition of G-equivariant maps

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Hence this triviality (where G_U or G_U/N is known as the *little group*):

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Expectation: X should now admit a system of imprimitivity based on $B = G(U)$, and hence be induced. That's rewarded:

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Let $U \in \mathfrak{n}^*/N$ *be an orbit such that* $H := G_U$ *is closed in* G*. Then* H *contains* N*, and* $Y \mapsto X = \mathrm{Ind}_H^G Y$ *defines a bijection between*

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The inverse map sends X *to the Kazhdan-Kostant-Sternberg reduced space of* X *at* U*: the quotient of* $\Phi(\cdot)_{|\mathfrak{n}}^{-1}(U)$ *by its characteristic foliation. Moreover*

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Obstruction step

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More details. $\tilde{\mathrm{U}}$ and $\tilde{\mathrm{G}}_{\mathrm{U}}$ are built as follows: Fix a $c \in \mathrm{U}$ once and for all, and write $\mathrm{N}_c^{\mathrm{o}}$ for the identity component of the stabilizer N_c.

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 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta(\mathrm{N}_c^{\mathrm{o}}) & \longrightarrow & \Delta(\mathrm{N}_c) & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta(\mathrm{N}_c^{\mathrm{o}}) & \longrightarrow & \mathrm{N} \times \mathrm{G}_c & \longrightarrow & \tilde{\mathrm{G}}_{\mathrm{U}} & \longrightarrow & 1 \\
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1 & \longrightarrow & \Delta(\mathrm{N}_c^{\mathrm{o}}) & \longrightarrow & \mathrm{N} \rtimes \mathrm{G}_c & \xrightarrow{\tilde{\pi}} & \tilde{\mathrm{G}}_{\mathrm{U}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
 & & 1 & \longrightarrow & \mathrm{G}_{\mathrm{U}} & \longrightarrow & \mathrm{G}_{\mathrm{U}} & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Obstruction step

More details. $\tilde{U}$ and $\tilde{G}_U$ are built as follows: Fix a $c \in U$ once and for all, and write N_c^o for the identity component of the stabilizer N_c.

- $\tilde{U} \to U$ is the covering $N/N_c^o \to N/N_c$ with group $\Gamma = N_c/N_c^o$.
- $\tilde{G}_U$ is defined by the middle row of the following diagram, where $\Delta(k) = (k^{-1}, k)$ and $\pi(n, l) = nl$:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta(N_c^o) & \longrightarrow & \Delta(N_c) & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta(N_c^o) & \longrightarrow & N \rtimes G_c & \xrightarrow{\tilde{\pi}} & \tilde{G}_U & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
 & & 1 & \longrightarrow & G_U & \longrightarrow & G_U & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

Synopsis

Putting the last 3 theorems together, we get:

Theorem (Z.)

Let (X, σ, Φ) *be a homogeneous hamiltonian* G*-space and* $N \subset G$ *a closed normal subgroup. Then* $\Phi(X)$ *sits above a* G*-orbit* $G(U)$ *in* $\mathfrak{n}^*/N$*. If the stabilizer* G_U *is closed and* $U = N(c)$ *has Mackey obstruction* $[\theta]$*, then there is a unique hamiltonian* $(\tilde{G}_U/N, [-\theta])$*-space* (V, ω, φ) *such that*

$$X = \mathrm{Ind}_{G_U}^{G}(\tilde{U} \times_{\Gamma} V)$$

where $\tilde{U} = N/N_c^o$ *is the covering of* U *with group* $\Gamma = N_c/N_c^o$*. Every homogeneous hamiltonian* $(\tilde{G}_U/N, [-\theta])$*-space* V *arises in this way.*

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Contact imprimitivity theorem

Trying to mimick Mackey analysis for ***nonabelian*** normal subgroups forced us to remember that unitary representations really correspond to ***prequantum*** G-spaces $(\tilde{X}, \alpha)$.

These are principal circle bundles $P : \tilde{X} \to X$ with G-action preserving a connection 1-form α. The G-action, 2-form $d\alpha$, and 'contact' moment map $\langle \tilde{\Phi}(\tilde{x}), Z\rangle = \alpha(Z(\tilde{x}))$ descend and make X a hamiltonian G-space.

Suppose $(\tilde{Y}, \beta)$ is a prequantum H-space over Y. Then $(T^*G \times \tilde{Y}, \theta + \beta)$ is a prequantum $G \times H$-space over $T^*G \times Y$. Reducing by H produces an ***induced*** prequantum G-space $(\mathrm{Ind}_H^G \tilde{Y}, \alpha_{\mathrm{ind}})$ over $\mathrm{Ind}_H^G Y$ admitting a transitive ***system of imprimitivity***: a G-invariant, commutative Lie algebra $\tilde{\mathfrak{f}}_{\mathrm{ind}}$ of complete α-preserving vector fields. In fact $\tilde{\mathfrak{f}}_{\mathrm{ind}} \simeq \mathfrak{f}_{\mathrm{ind}}$.

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Let $(\tilde{X}, \alpha)$ be a prequantum G-space. The following are equivalent:

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End!